JMO mentoring scheme

April 2011 paper - answers

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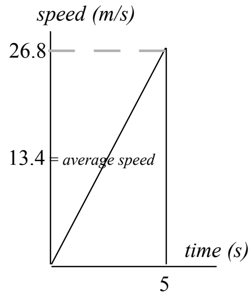
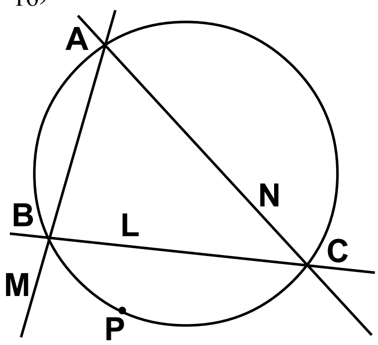
A1 Ans : 67 m (to nearest metre)
The average speed is 30 mph = 30 × 1609 ÷ 3600 m/s.
Note that the calculation is the same as finding the area of the triangle on the speed-time graph. This principle can be used whatever the shape of the speed-time graph.



B2 Each bracketed section exceeds $^1/_2$.
For example in $(^1/_5 + ^1/_6 + ^1/_7 + ^1/_8)$, each fraction is at least $^1/_8$.
Thus $1 + ^1/_2 + (^1/_3 + ^1/_4) + (^1/_5 + ^1/_6 + ^1/_7 + ^1/_8) + (^1/_9 + ^1/_{10} + ... + ^1/_{15} + ^1/_{16})$
$> 1 + ^1/_2 + (^1/_4 + ^1/_4) + (^1/_8 + ^1/_8 + ^1/_8 + ^1/_8) + (^1/_{16} + ^1/_{16} + ... + ^1/_{16} + ^1/_{16}) = 3$.

B3 Ans : AMPN, BMPL and CNLP.
The diameters of the circles are AP, BP and CP because
∠AMP, ∠ANP, ∠BMP, ∠BLP, ∠CLP and ∠CNP are all 90°.



B4 $(\sqrt{2} - 1)(\sqrt{2} + 1) = \sqrt{2}\sqrt{2} - 1\sqrt{2} + 1\sqrt{2} - 1 = 2 - 1 = 1$.
If $x = 1/(\sqrt{2} + 1)$, $x = \sqrt{2} - 1$ so $\sqrt{2} = 1 + x$.
Adding 1 to both sides gives $\sqrt{2} + 1 = 2 + x$.
Hence $1/(\sqrt{2} + 1) = 1/(2 + x)$ so $x = 1/(2 + x)$.

B5 One way to prove this is by placing a point inside the polygon which will produce n triangles.
If this point is moved to one of the sides, a triangle disappears.
If the point is moved to one of the vertices, two triangles disappear.

C6 Assume that c is even. Then c^2 is even.
This can happen if both a^2 and b^2 are even, i.e. a and b are even.
This contradicts the condition that the highest common factor is 1.
If both a^2 and b^2 are odd, c^2 is even but has no integer square root.
This is because $(2x + 1)^2 + (2y + 1)^2$ when multiplied out can be considered as a multiple of 4 plus 2 whereas every even square number is a multiple of 4.

C7 Ans : $\binom{2}{1} + \binom{2}{1} = \binom{3}{2}$
The first two prefixes are the same, the third one more.
The second and third suffixes are one more than the first.

You will probably get as far as showing that **1** comes from **1** above, **5** comes from **1** + **4** above, **6** from **3** + **3** and **1** from **1** + **0**. Since **1** + **4** + **3** + **0** is the previous Fibonacci number and **1** + **3** + **1** is the Fibonacci number before that, we have the rule associated with the Fibonacci numbers. The proof could be made more formal using the notation that has been introduced.

columns	0	1	2	3	4	5	6	7
row 0	1	0	0	0	0	0	0	0
row 1	1	1	0	0	0	0	0	0
row 2	1	2	**1**	**0**	0	0	0	0
row 3	1	**3**	**3**	**1**	0	0	0	0
row 4	**1**	**4**	**6** +	4	1	0	0	0
row 5	**1**	**5** +	10	10	5	1	0	0
row 6	**1** +	6	15	20	15	6	1	0