JMO mentoring scheme

April 2011 paper

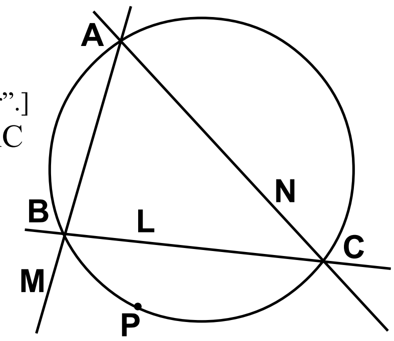
Supported by ManGroupplc Charitable Trust

'A' questions in each paper are meant to be straightforward - 'B' and 'C' questions more difficult.
Hints are included upside down at the bottom of the page. Fold this back and look at them when you need.

A1 A powerful sports car accelerates from 0 to 60 mph in 5 seconds so that the speed-time graph is a straight line. Using *distance = average speed × time*, find in metres the distance it travels in 5 seconds. 1 mile = 1609 m, 1 hour = 3600 seconds.

B2. Show that $1 + {}^1/_2 + ({}^1/_3 + {}^1/_4) + ({}^1/_5 + {}^1/_6 + {}^1/_7 + {}^1/_8) + ({}^1/_9 + {}^1/_{10} + \ldots + {}^1/_{15} + {}^1/_{16}) > 3$.

B3 Let A, B, C and P be distinct points on the circumference of a circle, with P lying between B and C, as shown in the diagram. A straight line is drawn from P meeting BC at L so that PL is perpendicular to BC.
[This is called the "perpendicular from P to BC" and L is the "foot of the perpendicular".]
Likewise M and N are the feet of the perpendiculars from P to AB and AC respectively The line AB or AC is extended as necessary.
Find, giving reasons, three quadrilaterals apart from ABCP whose vertices lie on circles.



B4 Show that $(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$.
If $x = 1/(\sqrt{2} + 1)$, prove that $\sqrt{2} = 1 + x$ and that $x = 1/(2 + x)$.

B5 Prove that the minimum number of triangles into which a convex n-sided polygon can be dissected is $n - 2$.

C6 a, b and c are integers with no common factor higher than 1 satisfying the equation $c^2 = a^2 + b^2$. Prove that c must be an odd number.

C7 Pascal's triangle can be laid out :

columns	0	1	2	3	4	5	6	7
row 0	1	0	0	0	0	0	0	0
row 1	1	1	0	0	0	0	0	0
row 2	1	***2***	***1***	0	0	0	0	0
row 3	1	3	***3***	**1**	0	0	0	0
row 4	1	4	**6** +	4	1	0	0	0
row 5	1	**5** +	10	10	5	1	0	0
row 6	**1** +	6	15	20	**15**	6	1	0

The advantage of this method is that each element can be referred to by its position. Thus we write

$$\binom{6}{4} = 15 \quad \text{(example shown in green/large bold).}$$

It has the property that two adjacent elements in a row add to the number in the next row underneath the right element (shown in red/bold italic). Write this example using the position notation.
Explain why adding along a diagonal from the left upwards (as example shown in blue/bold upright) produces a Fibonacci number.

** Reminder : notes published with the October paper can be found separately on the UKMT webpage following the links : www.ukmt.org.uk > Mentoring > Junior > Ideas in number theory, ... (pdf file).

Hints :

3. It's all to do with right angles formed on a diameter in a semi-circle : see note E.
4. Remember $a^2 - b^2 = (a - b)(a + b)$? [See note B]
5. Consider cutting up the polygon into triangles using an extra point **inside** the polygon.
6. Consider why c can **not** be an even number. Can two odd square numbers add to an even square number?
7. Careful study will show why it is sensible to start counting from row 0 and column 0. Writing out more examples like the one with the red numbers will give you confidence in using the notation.