# British Mathematical Olympiad

## Round 1

2011 to 2014

United Kingdom Mathematics Trust

United Kingdom Mathematics Trust

# British Mathematical Olympiad

## Round 1 : Friday, 2 December 2011

**Time allowed** $3\frac{1}{2}$ hours.

**Instructions**

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
- *Staple all the pages neatly together in the top* left *hand corner.*
- *To accommodate candidates sitting in other time-zones, please do not discuss the paper on the internet until 8am GMT on Saturday 3 December.*

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

# 2011/12 British Mathematical Olympiad
# Round 1: Friday, 2 December 2011

1. Find all (positive or negative) integers $n$ for which $n^2 + 20n + 11$ is a perfect square. *Remember that you must justify that you have found them all.*

2. Consider the numbers $1, 2, \ldots, n$. Find, in terms of $n$, the largest integer $t$ such that these numbers can be arranged in a row so that all consecutive terms differ by at least $t$.

3. Consider a circle $S$. The point $P$ lies outside $S$ and a line is drawn through $P$, cutting $S$ at distinct points $X$ and $Y$. Circles $S_1$ and $S_2$ are drawn through $P$ which are tangent to $S$ at $X$ and $Y$ respectively. Prove that the difference of the radii of $S_1$ and $S_2$ is independent of the positions of $P$, $X$ and $Y$.

4. Initially there are $m$ balls in one bag, and $n$ in the other, where $m, n > 0$. Two different operations are allowed:

   a) Remove an equal number of balls from each bag;
   b) Double the number of balls in one bag.

   Is it always possible to empty both bags after a finite sequence of operations?

   Operation b) is now replaced with

   b′) Triple the number of balls in one bag.

   Is it now always possible to empty both bags after a finite sequence of operations?

5. Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers.

6. Let $ABC$ be an acute-angled triangle. The feet of the altitudes from $A, B$ and $C$ are $D$, $E$ and $F$ respectively. Prove that $DE + DF \leq BC$ and determine the triangles for which equality holds.

   *The altitude from $A$ is the line through $A$ which is perpendicular to $BC$. The foot of this altitude is the point $D$ where it meets $BC$. The other altitudes are similarly defined.*

United Kingdom Mathematics Trust

# British Mathematical Olympiad

**Round 1 : Friday, 30 November 2012**

**Time allowed** $3\frac{1}{2}$ hours.

**Instructions**

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
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Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

# 2012/13 British Mathematical Olympiad
## Round 1: Friday, 30 November 2012

1. Isaac places some counters onto the squares of an 8 by 8 chessboard so that there is at most one counter in each of the 64 squares. Determine, with justification, the maximum number that he can place without having five or more counters in the same row, or in the same column, or on either of the two long diagonals.

2. Two circles $S$ and $T$ touch at $X$. They have a common tangent which meets $S$ at $A$ and $T$ at $B$. The points $A$ and $B$ are different. Let $AP$ be a diameter of $S$. Prove that $B$, $X$ and $P$ lie on a straight line.

3. Find all real numbers $x, y$ and $z$ which satisfy the simultaneous equations $x^2 - 4y + 7 = 0$, $y^2 - 6z + 14 = 0$ and $z^2 - 2x - 7 = 0$.

4. Find all positive integers $n$ such that $12n - 119$ and $75n - 539$ are both perfect squares.

5. A triangle has sides of length at most 2, 3 and 4 respectively. Determine, with proof, the maximum possible area of the triangle.

6. Let $ABC$ be a triangle. Let $S$ be the circle through $B$ tangent to $CA$ at $A$ and let $T$ be the circle through $C$ tangent to $AB$ at $A$. The circles $S$ and $T$ intersect at $A$ and $D$. Let $E$ be the point where the line $AD$ meets the circle $ABC$. Prove that $D$ is the midpoint of $AE$.

United Kingdom Mathematics Trust

# British Mathematical Olympiad

**Round 1 : Friday, 29 November 2013**

**Time allowed** $3\frac{1}{2}$ hours.

**Instructions**

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United Kingdom Mathematics Trust

# 2013/14 British Mathematical Olympiad
# Round 1: Friday, 29 November 2013

1. Calculate the value of
$$\frac{2014^4 + 4 \times 2013^4}{2013^2 + 4027^2} - \frac{2012^4 + 4 \times 2013^4}{2013^2 + 4025^2}.$$

2. In the acute-angled triangle $ABC$, the foot of the perpendicular from $B$ to $CA$ is $E$. Let $l$ be the tangent to the circle $ABC$ at $B$. The foot of the perpendicular from $C$ to $l$ is $F$. Prove that $EF$ is parallel to $AB$.

3. A number written in base 10 is a string of $3^{2013}$ digit 3s. No other digit appears. Find the highest power of 3 which divides this number.

4. Isaac is planning a nine-day holiday. Every day he will go surfing, or water skiing, or he will rest. On any given day he does just one of these three things. He never does different water-sports on consecutive days. How many schedules are possible for the holiday?

5. Let $ABC$ be an equilateral triangle, and $P$ be a point inside this triangle. Let $D, E$ and $F$ be the feet of the perpendiculars from $P$ to the sides $BC$, $CA$ and $AB$ respectively. Prove that

   a) $AF + BD + CE = AE + BF + CD$ and
   b) $[APF] + [BPD] + [CPE] = [APE] + [BPF] + [CPD]$.

   *The area of triangle $XYZ$ is denoted $[XYZ]$.*

6. The angles $A, B$ and $C$ of a triangle are measured in degrees, and the lengths of the opposite sides are $a, b$ and $c$ respectively. Prove that
$$60 \leq \frac{aA + bB + cC}{a + b + c} < 90.$$

United Kingdom Mathematics Trust

# British Mathematical Olympiad

## Round 1 : Friday, 28 November 2014

**Time allowed** $3\frac{1}{2}$ hours.

**Instructions**

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
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United Kingdom Mathematics Trust

# 2014/15 British Mathematical Olympiad
# Round 1: Friday, 28 November 2014

1. Place the following numbers in increasing order of size, and justify your reasoning:
$$3^{3^4}, 3^{4^3}, 3^{4^4}, 4^{3^3} \text{ and } 4^{3^4}.$$
*Note that $a^{b^c}$ means $a^{(b^c)}$.*

2. Positive integers $p$, $a$ and $b$ satisfy the equation $p^2 + a^2 = b^2$. Prove that if $p$ is a prime greater than 3, then $a$ is a multiple of 12 and $2(p + a + 1)$ is a perfect square.

3. A hotel has ten rooms along each side of a corridor. An olympiad team leader wishes to book seven rooms on the corridor so that no two reserved rooms on the same side of the corridor are adjacent. In how many ways can this be done?

4. Let $x$ be a real number such that $t = x + x^{-1}$ is an integer greater than 2. Prove that $t_n = x^n + x^{-n}$ is an integer for all positive integers $n$. Determine the values of $n$ for which $t$ divides $t_n$.

5. Let $ABCD$ be a cyclic quadrilateral. Let $F$ be the midpoint of the arc $AB$ of its circumcircle which does not contain $C$ or $D$. Let the lines $DF$ and $AC$ meet at $P$ and the lines $CF$ and $BD$ meet at $Q$. Prove that the lines $PQ$ and $AB$ are parallel.

6. Determine all functions $f(n)$ from the positive integers to the positive integers which satisfy the following condition: whenever $a$, $b$ and $c$ are positive integers such that $1/a + 1/b = 1/c$, then
$$1/f(a) + 1/f(b) = 1/f(c).$$

# Introduction to the problems and full solutions

The 'official' solutions are the result of many hours' work by a large number of people, and have been subjected to many drafts and revisions. The contestants' solutions included here will also have been redrafted several times by the contestants themselves, and also shortened and cleaned up somewhat by the editors. As such, they do not resemble the first jottings, failed ideas and discarded pages of rough work with which any solution is started.

Before looking at the solutions, pupils (and teachers) are encouraged to make a concerted effort to attack the problems themselves. Only by doing so is it possible to develop a feel for the question, to understand where the difficulties lie and why one method of attack is successful while others may fail. Problem solving is a skill that can only be learnt by practice; going straight to the solutions is unlikely to be of any benefit.

It is also important to bear in mind that solutions to Olympiad problems are not marked for elegance. A solution that is completely valid will receive a full score, no matter how long and tortuous it may be. However, elegance has been an important factor influencing our selection of contestants' answers.

# Solutions

# 2011

## BMO Round 1

**Problem 1** (Proposed by Dr Gerry Leversha.) *Find all (positive or negative) integers $n$ for which $n^2 + 20n + 11$ is a perfect square.* Remember that you must justify that you have found them all.

**Solution (Rafi Dover, King David High School):** Let $n^2 + 20n + 11 = a^2$ where $a$ is a positive integer. Then

$$\begin{aligned} n^2 + 20n + 11 &= a^2 \\ (n+10)^2 - 100 + 11 &= a^2 \\ (n+10)^2 &= a^2 + 89 \\ (n+10)^2 - a^2 &= 89 \end{aligned}$$

So solutions correspond to pairs of squares that differ by 89.

We claim that $\sum_{r=1}^{n}(2r-1) = n^2$ – the proof is by induction. When $n = 1$, this is clearly true, and if we suppose that $\sum_{r=1}^{k} = k^2$ then

$$\sum_{r=1}^{k+1}(2r-1) = \sum_{r=1}^{k}(2r-1) + 2k + 1 = (k+1)^2$$

and so our claim is true for all $n$.

So for positive integers $x$ and $y$,

$$x^2 - y^2 = \sum_{r=y+1}^{x}(2r-1)$$

So whenever we have a solution to the original equation, we have a set of consecutive positive odd integers that sum to 89. The sum of an even number of odd integers is even, and as 89 is odd this means that there must be an odd number of integers in the sum. But then there is a 'middle' integer in the list, which divides the sum. But as 89 is prime, the only way we can make 89 as a sum of consecutive odd integers is 89 itself.

So we must have $a = 44$ and hence $(n+10)^2 = 45^2$. This gives us two possibilities (corresponding to positive and negative square roots) for $n$: 35 and $-55$. We can check that these do both give us $n^2 + 20n + 11 = 44^2$.

**Problem 2** (Proposed by Dr Geoff Smith.) *Consider the numbers $1, 2, \ldots, n$. Find, in terms of $n$, the largest $t$ such that these numbers can be arranged in a row so that all consecutive terms differ by at least $t$.*

**Solution (Natalie Behague, Dartford Grammar School for Girls):** First consider the case when $n$ is even, $n = 2m$. Then the sequence

$$m, 2m, m-1, 2m-1, \ldots 1, m+1$$

has every consecutive term differing by $m$ or $m+1$. So we can take $t = m = \frac{n}{2}$. We can't have $t > m$ because the sequence would have to contain $m$, which is distance at most $m$ from all the other terms.

Now consider the case where $n$ is odd; $n = 2m+1$. We can take the sequence

$$2m+1, m+1, 1, m+2, 2, m+3, \ldots, 2m, m$$

which has every term differeing by $m$ or $m+1$. So we can take $t = m = \frac{n-1}{2}$. We can't have $t > m$ because $m+1$ differs by at most $m$ from each other term in the sequence.

The best value of $t$ is therefore

$$t = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

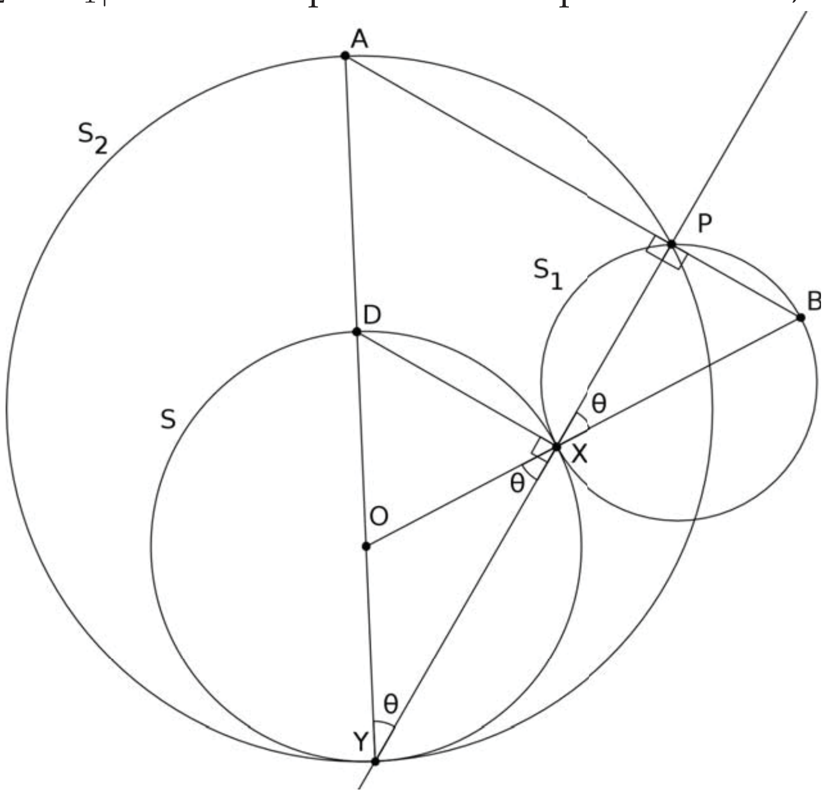
**Problem 3** (Proposed by Dr Gerry Leversha.) *Consider a circle $S$. The point $P$ lies outside of $S$ and a line is drawn through $P$, cutting $S$ at distinct points $X$ and $Y$. Circles $S_1$ and $S_2$ are drawn through $P$ which are tangent to $S$ at $X$ and $Y$ respectively. Prove that the difference of the radii of $S_1$ and $S_2$ is independent of the positions of $P$, $X$ and $Y$.*

The first solution given below makes elegant use of the fact that if two circles are tangent then there is an enlargement centered on the point of tangency taking one to the other. The second solution is similar at heart but instead uses several similar right-angled triangles.

**Solution 1 (Adam Goucher, Netherthorpe School):** Assume, without loss of generality, that $PX < PY$. Denote by $R$, $R_1$ and $R_2$ the radii of $S$, $S_1$ and $S_2$ respectively. $S_2$ and $S$ are then internally tangent at $Y$, so that there is an enlargement about $Y$ with positive scale factor $\frac{R_2}{R}$ sending $S$ to $S_2$. Similarly, there is an enlargement about $X$ with negative scale factor $-\frac{R_1}{R}$ sending $S$ to $S_1$. The first enlargement maps $XY$ to $PY$, so that $\frac{PY}{XY} = \frac{R_2}{R}$. The second enlargement maps $XY$ to $XP$, so that $\frac{PX}{XY} = \frac{R_1}{R}$. Subtracting these two equations gives

$$\frac{R_2 - R_1}{R} = \frac{PY - PX}{XY} = 1.$$

Therefore $|R_2 - R_1| = R$ is independent of the positions of $P$, $X$ and $Y$.




**Solution 2 (B.T.G. de Jager, Eton College):** Without loss of generality, assume that $PX < PY$. Let $O$ be the center of $S$, and let $R$, $R_1$ and $R_2$ be

as in the previous solution. Let $YO$ meet $S$ again at $D$ and $S_2$ again at $A$, and let $B$ be diametrically opposite $X$ on $S_1$. Then $B$, $X$ and $O$ are collinear because $S_1$ and $S$ are tangent, and $AY$ is a diameter of $S_2$ because $S_2$ and $S$ are tangent. Therefore, because the angle in a semicircle is $90°$, we have that $\angle DXY$, $\angle APY$, and $\angle XPB$ are all $90°$.

Let $\angle AYP = \theta$. Then, because $OX = OY$, triangle $OXY$ is isosceles and so $\angle PXB = \angle YXO = \angle OYX$. From right-angled triangle $APY$, we find that $2R_2 = \frac{PY}{\cos\theta}$. Similarly, from right-angled triangle $BPX$, we find that $2R_1 = \frac{PX}{\cos\theta}$. Subtracting these we get

$$2(R_2 - R_1) = \frac{PY - PX}{\cos\theta} = \frac{XY}{\cos\theta}.$$

But from right-angled triangle $DXY$ this is equal to $DY = 2R$. So $|R_2 - R_1| = R$, which is independent of the positions of $P$, $X$ and $Y$.

**Problem 4** (Proposed by Dr Gerry Leversha.) *Initially there are $m$ balls in one bag, and $n$ in the other, where $m, n > 0$. Two different operations are allowed:*

*a) Remove an equal number of balls from each bag;*

*b) Double the number of balls in one bag.*

*Is it always possible to empty both bags after a finite sequence of operations?*

*Operation b) is now replaced by*

*b′) Triple the number of balls in one bag*

*Is it now always possible to empty both bags after a finite sequence of operations?*

**Solution (Andrew Carlotti, Sir Roger Manwood's School):** We claim in the first case that it is always possible. If $m = n$ then we can just empty both bags. Otherwise, we may as well assume $m < n$. We can double the number of balls in the bag with $m$ balls repeatedly (if necessary) until there are at least $\frac{n}{2}$ balls in that bag. There are now $m'$ and $n$ (where $\frac{n}{2} \leq m' < n$) balls in the two bags. We can then remove $2m - n$ from each bag, at which point there are $n - m$ and $2(n - m)$ balls in each bag. Then we double the number of balls in the bag with fewer balls in it, and finally empty both bags.

With the modified operation, it is not always possible. Operations a) and b′) both alter the total number of balls by an even number, and so if there are an odd number of balls in total (say 2 balls in the first bag, and 1 in the second), we cannot empty them both.

**Problem 5** (Proposed by Robin Bhattacharyya) *Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers.*

The main difficulty with this problem is finding a way to work with the fact that we are looking for *positive integer* solutions, as any approach which attempts to solve the below equations without this cannot work. Most solutions given were variants on one of the two similar solutions shown here, where we 'sandwich' possible solutions between two consecutive integers.

**Solution 1 (Katya Richards, The School of St Helen and St Katharine):** Let the four consecutive integers by $b, b+1, b+2, b+3$. Then their product is

$$b(b+3)(b+1)(b+2) = (b^2+3b)(b^2+3b+2)$$

Let $c = b^2 + 3b$. Then this product is equal to $c(c+2)$. If this product were equal to the product of two consecutive integers, $a$ and $(a+1)$, then we would have $a(a+1) = c(c+2)$.

We clearly can't have $a = c$ or $a = c+1$, since $c(c+1) < c(c+2) < (c+1)(c+2)$. We also can't have $c < a$ because then also $(a+1) < (c+2)$ and so $a(a+1) < c(c+2)$. But we also can't have $a > (c+1)$ as then $(a+1) > (c+2)$ and so $a(a+1) > c(c+2)$. So there are no solutions to $a(a+1) = c(c+2)$, and hence we can't write the product of four consecutive positive integers as the product of two consecutive positive integers.

**Solution 2 (Oliver Feng, Eton College):** Suppose that

$$\begin{aligned} n(n+1) &= m(m+1)(m+2)(m+3) \\ &= (m+1)(m+2) \cdot m(m+3) \\ &= (m^2+3m+2)(m^2+3m) \\ &= (m^2+3m+1)^2 - 1 \end{aligned}$$

So setting $M = m^2 + 3m + 1$ we have $n(n+1) = M^2 - 1$. Then

$$M^2 = n(n+1) + 1 = n^2 + n + 1$$

However,

$$n^2 < n^2 + n + 1 < n^2 + 2n + 1 = (n+1)^2$$

for $n$ a positive integer. Then $M^2$ is a square number strictly between $n^2$ and $(n+1)^2$, which is clearly impossible. So no such $m$ and $n$ exist.

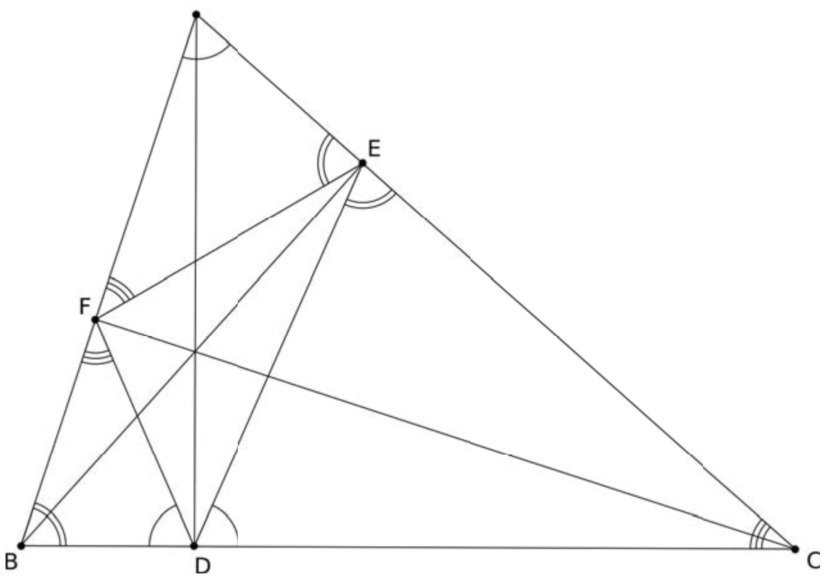
**Problem 6** (Proposed by Dr David Monk.) *Let $ABC$ be an acute-angled triangle. The feet of the altitudes from $A$, $B$ and $C$ are $D$, $E$ and $F$ respectively. Prove that $DE + EF \leq BC$ and determine the triangles for which equality holds.*

The altitude through $A$ is the line through $A$ which is perpendicular to $BC$. The foot of this altitude is the point $D$ where it meets $BC$. The other altitudes are similarly defined.

Most people who solved this problem used trigonometry to reduce the problem to an algebraic inequality, which could then be solved with a variety of methods – solution 1 is a nice example of this approach. It is also possible to give a more 'geometric' proof of the inequality, as in solution 3. Solution 2 is an interesting hybrid of the two approaches.

An angle-chase common to all three solutions is this: because $\angle AEB$ and $\angle ADB$ are right angled, $AEDB$ is cyclic with diameter $AB$. So $\angle EDC = 180 - \angle BDE = \angle BAE = \angle BAC$ by the fact that opposite angles in a cyclic quadrilateral add to $180°$. Similarly, $\angle FDC = \angle CAB$.

In all solutions, let $A$, $B$ and $C$ be the angles of triangle $ABC$ and let $a$, $b$ and $c$ be the lengths of $BC$, $CA$ and $AB$ respectively.




**Solution 1 (Yu Wan):** First we will compute $DE$ and $DF$. From right-angled triangles $BEC$ and $ADC$ we have that $CE = a\cos C$. By the sine rule in triangle $CED$, $DE = \sin C \frac{CE}{\sin EDC} = \sin C \frac{a \cos C}{\sin A}$ by the previous sentence and the angle chase in the introduction. By the sine rule in triangle $ABC$, $\frac{a}{\sin A} = \frac{c}{\sin C}$ so that $DE = c\cos C$. Similarly, $DF = b\cos B$.

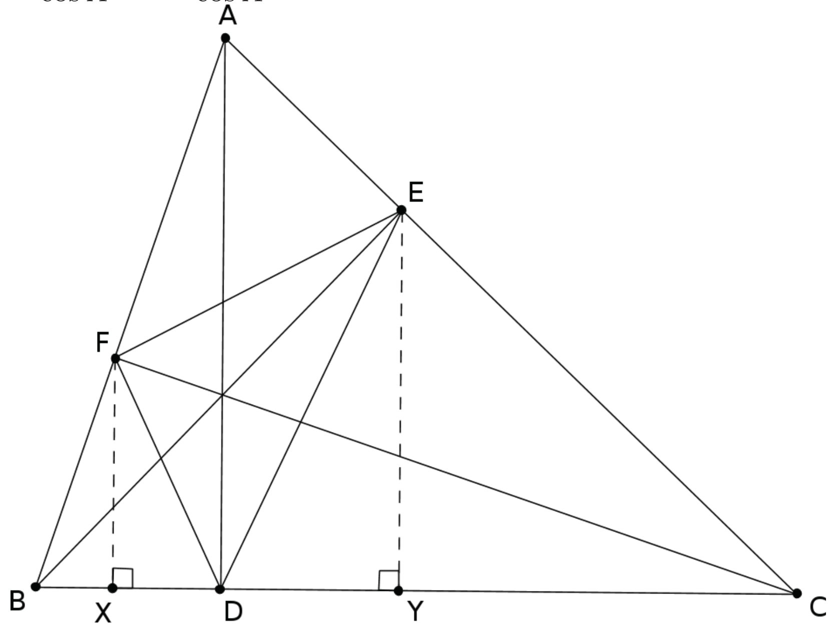
So we have $DE + DF = c\cos C + b\cos B$, while $BC = BD + DC = c\cos B + b\cos C$. Therefore

$$\begin{aligned} DE + DF - BC &= c\cos C + b\cos B - c\cos B - b\cos C \\ &= (c-b)(\cos C - \cos B). \end{aligned}$$

If $c > b$, then $C > B$, so $\cos C < \cos B$ and therefore $DE + DF < BC$. If $c = b$ then $DE + DF = BC$. If $c < b$, then $C < B$, so $\cos C > \cos B$ and therefore $DE + DF < BC$.

We have shown that $DE + DF \leq BC$ with equality if, and only if, $b = c$.

**Solution 2 (Joshua Lam, The Leys School):** Let $X$ and $Y$ be the feet of the perpendiculars from $F$ and $E$ to $BC$, respectively. As in the introduction, we have $\angle FDX = A$ and $\angle EDY = A$. So $DF = \frac{DX}{\cos A}$ and $DE = \frac{DY}{\cos A}$, so that $DF + DE = \frac{DX+DY}{\cos A} = \frac{XY}{\cos A}$. So it suffices to show that $XY \leq a \cos A$.

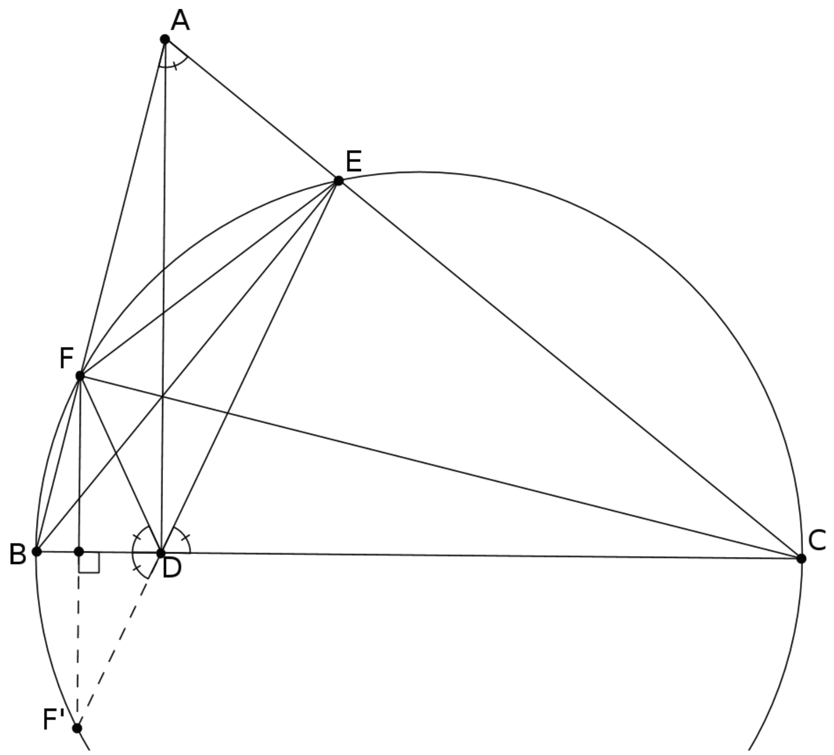



But $XY$ is the orthogonal projection of $FE$ onto the side $BC$, so that $FE \geq XY$ (essentially, because the side opposite the right angle is the longest side of a right-angled triangle). Equality holds whenever $FE$ is parallel to $XY$. We will show that $FE = a \cos A$, which gives $XY \leq a \cos A$.

This calculation is exactly the same as the calculation of $DE$ in the previous solution, and the result is that $FE = a \cos A$ as required.

Equality holds if and only if $FE$ is parallel to $XY$. This happens if and only if $\angle AFE = \angle ABC$. But the cyclic quadrilateral $BFEC$ gives that $\angle AFE = \angle ACB$. So equality occurs if, and only if, $\angle ABC = \angle ACB$ i.e. that $AB = AC$.

**Solution 3 (by the markers):** As before, we have that $\angle EDC = \angle FDB = A$. This means that if we reflect $F$ in the line $BC$ to get $F'$, then $\angle F'DB = \angle EDC$, so that $F'DE$ is a straight line. But because $\angle BF'C = \angle BEC = 90°$, $F'BEC$ is cyclic with diameter $BC$. The diameter is the greatest distance between two points on the circumference of a circle, so that $BC \geq F'E = F'D + DE = DF + DE$, as required.




Equality holds if, and only if, $EF'$ is a diameter of $F'BEC$, which in turn happens if, and only if, $\angle F'CE = 90°$. But $\angle F'CE = \angle F'CB + \angle ECB = \angle FCB + \angle ECB = 90 - B + C$. So equality occurs if, and only if, $B = C$ i.e. $AB = AC$.

# Solutions
# 2012

## BMO Round 1

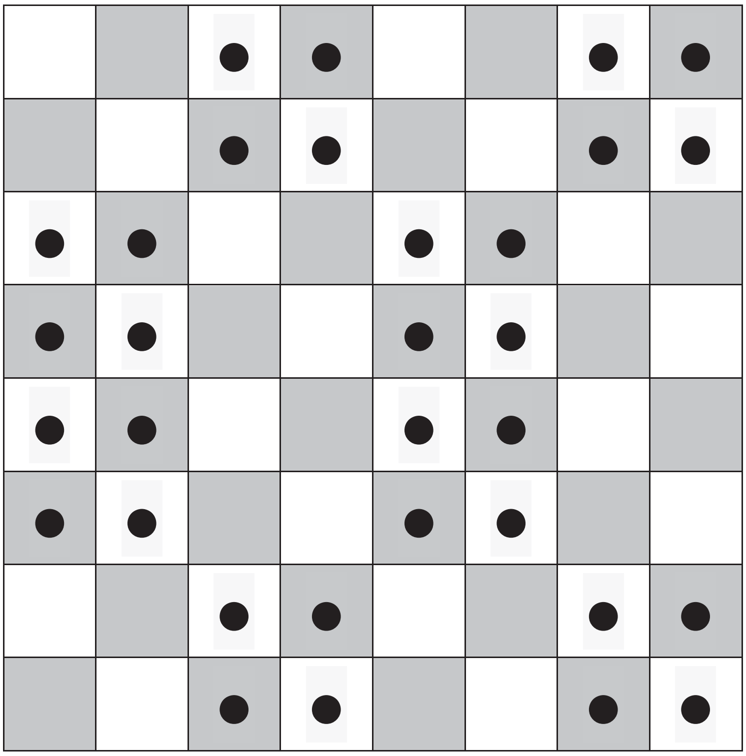
**Problem 1** (Proposed by Dr Jeremy King.) *Isaac places some counters into the squares of an $8 \times 8$ chessboard so that there is at most one counter in each of the 64 squares. Determine, with justification, the maximum number that he can place without having five or more counters in the same row, or in the same column, or in either of the two long diagonals.*

The key to this problem is to realise that, if we focus only on the constraint that there can be only four counters in each row, the maximum number of counters we can put on the board is 32. (Of course, we could also do the same by considering only the columns). This can be attained in a variety of ways.

**Solution by Abigail Hayes, Stretford Grammar School:** We know that there must be fewer than 5 counters in each row, column and long diagonal, so the maximum in each is 4 counters.

If we had more than 32 counters, then there would be at least 5 in one row, which cannot happen. Hence, there can be at most 32 counters.

Here is an example which attains this bound:

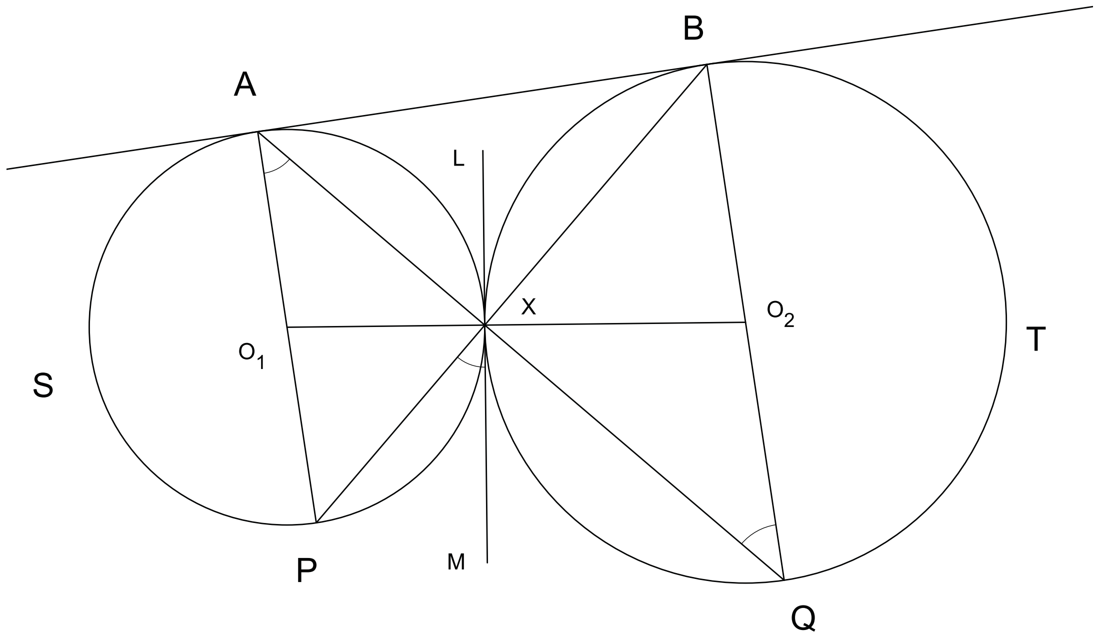


**Problem 2** (Proposed by Richard Freeland.) *Two circles $S$ and $T$ touch at $X$. They have a common tangent which meets $S$ at $A$ and $T$ at $B$. The points $A$ and $B$ are different. Let $AP$ be a diameter of $S$. Prove that $B, X$ and $P$ lie on a straight line.*

There are a few different ways to approach this problem, mostly using the alternate segment theorem and other standard circle facts. Whilst a good diagram is always helpful, it is important not to accidentally *assume* that $PXB$ is a straight line as part of the proof, just because it looks that way in the diagram!

**Solution 1 by Harry Metrebian, Winchester College:** Let the centres of circles $S$ and $T$ be $O_1$ and $O_2$ respectively; let the other end of the diameter of $T$ through $B$ be $Q$ and let the common tangent to both circles at $X$ pass through a point $L$ on the same side of $X$ as the line $AB$ and a point $M$ on the other side of $X$.

We know that $O_1XO_2$ is a straight line as the lines $O_1X$ and $O_2X$ are both perpendicular to the tangent at $X$.




Since tangents and radii meet at right angles,

$$\angle PAB = \angle ABQ = 90^\circ.$$

Therefore the lines $AP$ and $BQ$ are parallel. So, by alternate angles,

$$\angle PO_1X = \angle BO_2X.$$

Since the angle subtended by a chord at the centre of a circle is twice that subtended by the same chord at the circumference,

$$\angle PAX = \frac{1}{2}\angle PO_1X = \frac{1}{2}\angle BO_2X = \angle BQX.$$

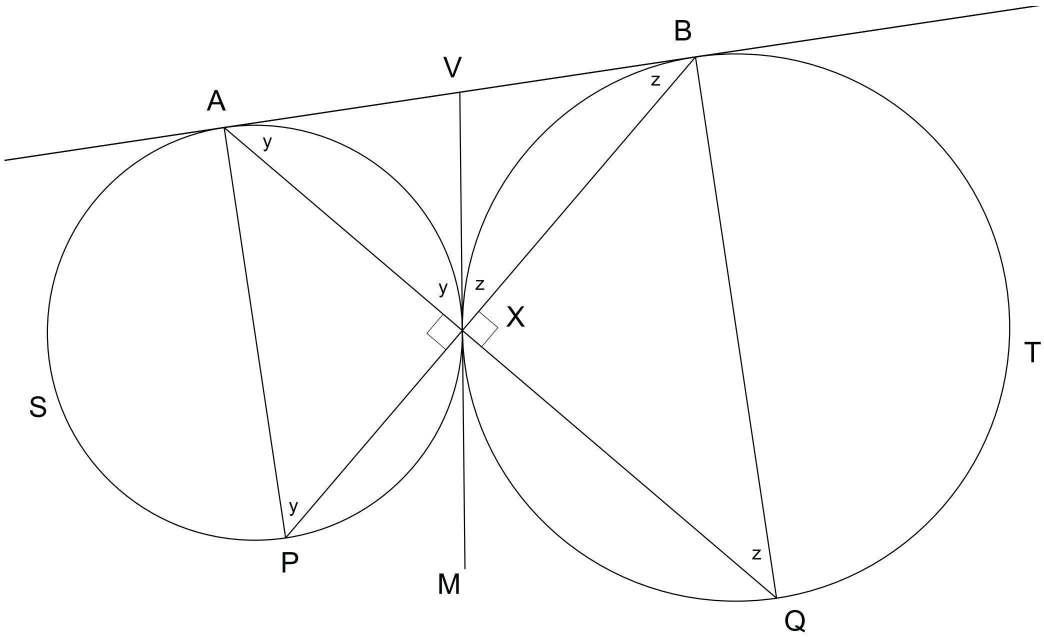
By the alternate segment theorem,

$$\angle PXM = \angle PAX = \angle BQX = \angle BXL.$$

Since $LXM$ is a straight line, $BXP$ is also a straight line by the converse of the vertically opposite angles theorem.

**Solution 2 by Aatreyee Mimi Das, Heckmondwike Grammar School:** Let the other end of the diameter of $T$ through $B$ be $Q$. Let the tangent to circles $S$ and $T$ at $X$ meet $AB$ at $V$, and let $M$ be a point on $XV$ the other side of $X$ from $V$.

Note that $\angle PXA = 90°$ because $AP$ is a diameter. Similarly $\angle BXQ = 90°$. Let $\angle APX = y$. Then by the alternate segment theorem, $\angle BAX = y$. Also let $\angle XQB = z$; then $\angle XBA = z$ also by the alternate segment theorem.




Using the alternate segment theorem again, it must be the case that $\angle AXV = y$ and $\angle VXB = z$. Then $\angle AXB = y + z$. But then the sum of the angles in triangle $BAX$ is

$$\begin{aligned} &\angle XBA + \angle BAX + \angle AXB \\ &= z + y + (y + z) \\ &= 2(y + z). \end{aligned}$$

Since this must be $180°$, it follows that $\angle AXB = y + z = 90°$. Therefore

$$\angle PXB = \angle PXA + \angle AXB = 90° + 90° = 180°,$$

and so $PXB$ is a straight line.

**Problem 3** (Proposed by Dr David Monk.) *Find all real numbers $x, y$ and $z$ which satisfy the simultaneous equations $x^2 - 4y + 7 = 0$, $y^2 - 6z + 14 = 0$ and $z^2 - 2x - 7 = 0$.*

It might not seem obvious where to start with this curious looking set of simultaneous equations, but there turns out to be a useful trick that leads to the solution. Most successful solutions used the method below.

**Solution by the markers:** Sum all three equations, to yield

$$x^2 - 4y + 7 + y^2 - 6z + 14 + z^2 - 2x - 7 = 0.$$

This can be rearranged to form

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 0.$$

Since all squares are nonnegative, this can only happen if all three terms are equal to 0. Therefore, if there is any solution, it must be that $x = 1, y = 2$ and $z = 3$.

We must check against the *original* equations to see if this is indeed a solution, and it turns out that $x = 1, y = 2, z = 3$ does satisfy them. Therefore this is the unique solution.

**Problem 4** (Proposed by Richard Freeland.) *Find all positive integers $n$ such that $12n - 119$ and $75n - 539$ are both perfect squares.*

The key in this question is to somehow reduce the problem to a finite set of cases which we can then check each of. A typical way of doing this is to find some factorisation of an integer, as the solution below does.

**Solution by Alex Harris, Perse School:** Firstly, let $a^2 = 12n - 119$ and $b^2 = 75n - 539$.

We have $n = \dfrac{a^2 + 119}{12}$, from the definition of $a$. So we can use the definition of $b$ to derive

$$\frac{75(a^2 + 119)}{12} - 539 = b^2,$$
$$25(a^2 + 119) - 4 \times 539 = 4b^2,$$

and hence

$$25a^2 - 4b^2 = -819.$$

This can be factorised to give

$$(2b - 5a)(2b + 5a) = 819.$$

So we know that the integers $2b-5a$ and $2b+5a$ multiply together to give $819$, so, since we can assume that $a$ and $b$ are nonnegative, we have a limited set of possibilities for $(2b - 5a, 2b + 5a)$: $(1, 819)$, $(3, 273)$, $(7, 117)$, $(9, 91)$, $(13, 63)$ and $(21, 39)$.

Only the second, third and fifth give integer values for $a$, so $(a, b)$ can be $(27, 69)$, $(11, 31)$ or $(5, 19)$.

We can substitute into the formula for $n$ above to determine its possible values. The first does not give an integer, so we are left with $n = 20$ or $n = 12$. We can easily check that both these values of $n$ work.

**Problem 5** (Proposed by Dr Jeremy King.) *A triangle has sides of length at most* 2, 3 *and* 4 *respectively. Determine, with proof, the maximum possible area of the triangle.*

Whilst there are some very neat, short solutions to this problem, it is important to carefully justify why the given triangle is maximal.

**Solution 1 by Gavin O'Connell, Bristol Grammar School:** Let $\theta$ be the largest angle of the triangle. Thus, $\theta$ is the angle between the two shorter sides, of lengths $a$ and $b$.

The area of the triangle is given by $A = \frac{1}{2}ab\sin\theta$, where $a \leq b$ are the shorter sides. But we have

$$a \leq 2,$$
$$b \leq 3,$$

and

$$\sin\theta \leq 1.$$

Hence $A \leq 3$, with equality when the triangle is right angled with legs 2 and 3. This gives a hypotenuse of length $\sqrt{13} < 4$, which is valid.

Thus, the maximum area is 3.

**Solution 2 by Warren Li, Fulford School:** If the sides of the triangle are of lengths $a$, $b$ and $c$, and the circumradius is $R$, then it is well known that the area of the triangle is $A = \frac{abc}{4R}$.

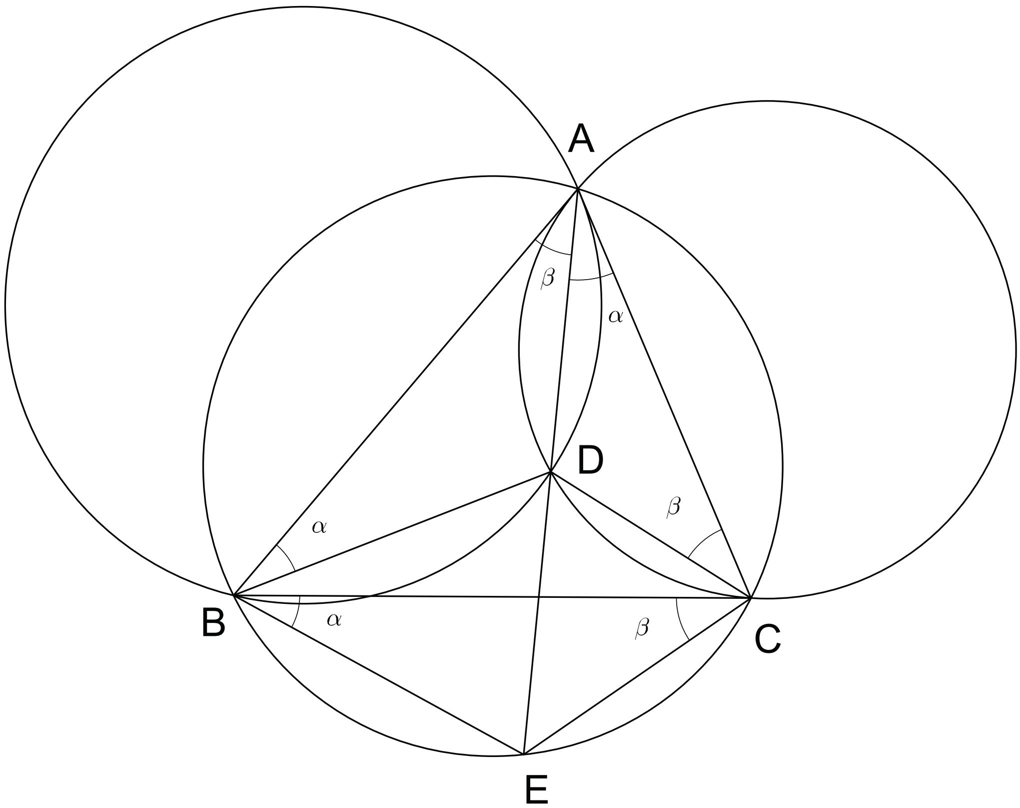
Let $a \leq b \leq c$, without loss of generality. We have $c \leq 2R$, $a \leq 2$ and $b \leq 3$, so $A \leq \frac{ab}{2} \leq 3$.

The same example from before works.

**Problem 6** (Proposed by Dr Gerry Leversha.) *Let $ABC$ be a triangle. Let $S$ be the circle through $B$ tangent to $CA$ at $A$ and let $T$ be the circle through $C$ tangent to $AB$ at $A$. The circles $S$ and $T$ intersect at $A$ and $D$. Let $E$ be the point where the line $AD$ meets the circle $ABC$. Prove that $D$ is the midpoint of $AE$.*

There are a wide variety of ways to approach this problem. Some students used a similar triangles argument along the lines of the first solution below; some other methods involved considering the centres of circles $S$ and $T$, as in the second solution. Another method is to observe that the circumcentre of $ABC$ lies on the circle through $B, D, C$, and extend the line $ADE$ to meet this circle again.

**Solution 1 by Oliver Feng, Eton College:** Let $\angle ABD = \alpha$ and $\angle DCA = \beta$.



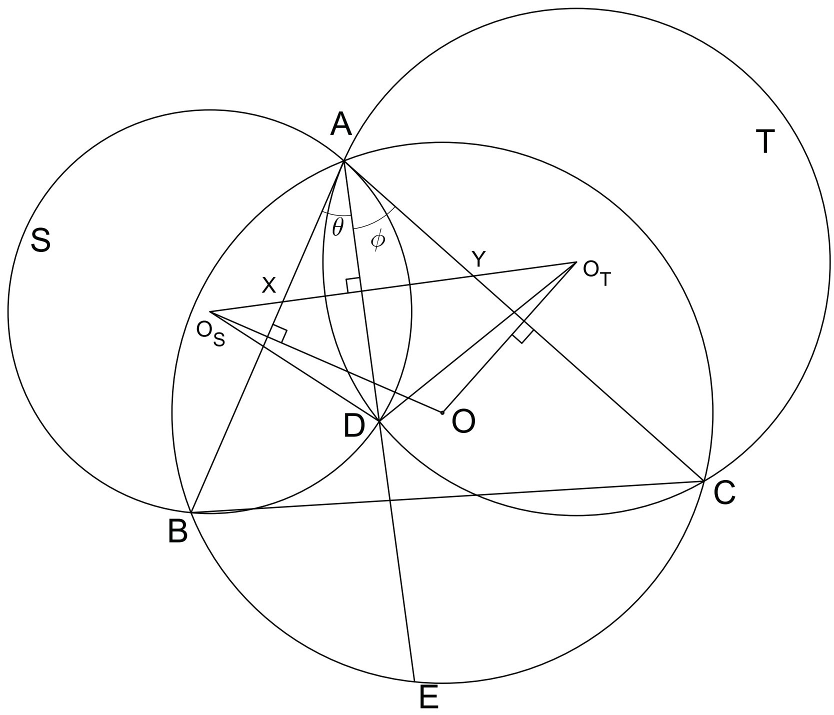

Then $\angle CAD = \angle ABD = \alpha$ and $\angle DAB = \angle DCA = \beta$, by the alternate segment theorem. So $\triangle ABD$ is similar to $\triangle CAD$.

Also, $\angle CBE = \angle CAE = \alpha$ and $\angle ECB = \angle EAB = \beta$ by the theorem of angles in the same segment. Similarly $\angle BED = \angle BEA = \angle BCA$. Further, $\angle DBE = \angle CBE + \angle DBC = \angle ABD + \angle DBC = \angle ABC$. So $\triangle DBE$ and $\triangle ABC$ are similar.

Therefore, $\frac{AB}{BD} = \frac{AC}{AD}$ and so $AD = \frac{AC \times BD}{AB}$. Also, $\frac{DB}{DE} = \frac{AB}{AC}$, and so $DE = \frac{AC \times BD}{AB} = AD$.

So $D$ is the midpoint of $AE$.

**Solution 2 by Ian Fan, Dr Challoner's Grammar School** Denote the centres of circles $S$ and $T$ respectively by $O_S$ and $O_T$, and let $\angle DAB = \theta$, $\angle CAE = \phi$.




We can see that

$$\angle O_T O_S O = 90^\circ - \angle BXO_S = \angle DAB = \theta$$

and

$$\angle OO_T O_S = 90^\circ - \angle O_T YC = \angle CAE = \phi.$$

Then

$$\angle O_T O_S D = 90^\circ - \angle O_S DA = 90^\circ - \angle DAO_S = \phi$$

since the line $AC$ is tangent to circle $S$. Similarly, $\angle DO_T O_S = \theta$. Combining this with what we deduced above,

$$\angle O_T O_S O = \angle DO_T O_S$$

and

$$\angle OO_TO_S = \angle O_TO_SD.$$

Therefore the two triangles $\triangle O_TO_SD$ and $\triangle O_SO_TO$ share the side $O_SO_T$ and the two angles at $O_S$ and $O_T$ (but the other way round). So one is a reflection of the other. Consider this reflection. It maps $O_S$ to $O_T$ and vice versa, so it must be a reflection in a line perpendicular to $O_SO_T$. Since $D$ maps to $O$ (and vice versa), this line must also be perpendicular to the line $DO$. So $DO$ is parallel to $O_SO_T$.

Since $O_SO_T$ is perpendicular to the line $ADE$, it must also be the case that $DO$ is perpendicular to $ADE$. As $\triangle AOE$ is isosceles with $AO = EO$, this implies that $D$ is the midpoint of $AE$.

# Solutions
# 2013

## BMO Round 1

**Problem 1** (Proposed by Andrew Jobbings) *Calculate the value of*

$$\frac{2014^4 + 4 \times 2013^4}{2013^2 + 4027^2} - \frac{2012^4 + 4 \times 2013^4}{2013^2 + 4025^2}.$$

The key to this problem was to realise that one could write the whole expression in terms of just one 'variable', 2013 (or, equivalently, 2012 or 2014). Once the expression is in an algebraic form it is much more natural to expand out the terms and simplify – we present two ways in which the resulting expression can be shown to vanish.

**Solution 1 by Eoghan McDowell, Silverdale School:** Let $a = 2013$. Then the given expression can be written as

$$\begin{aligned}
&\frac{(a+1)^4 + 4a^4}{a^2 + (2a+1)^2} - \frac{(a-1)^4 + 4a^4}{a^2 + (2a-1)^2} \\
=&\frac{a^4 + 4a^3 + 6a^2 + 4a + 1 + 4a^4}{a^2 + 4a^2 + 4a + 1} - \frac{a^2 - 4a^3 + 6a^2 - 4a + 1 + 4a^2}{a^2 + 4a^2 - 4a + 1}
\end{aligned}$$

which can then be simplified to

$$\begin{aligned}
&\frac{5a^4 + 4a^3 + 6a^2 + 4a + 1}{5a^2 + 4a + 1} - \frac{5a^4 - 4a^3 + 6a^2 - 4a + 1}{5a^2 - 4a + 1} \\
=&\frac{a^2(5a^2 + 4a + 1) + 5a^2 + 4a + 1}{5a^2 + 4a + 1} - \frac{a^2(5a^2 - 4a + 1) + 5a^2 - 4a + 1}{5a^2 - 4a + 1} \\
=&(a^2 + 1) - (a^2 + 1)
\end{aligned}$$

and so the answer is 0.

**Solution 2 by Rory Boath, The Grange School, Hartford (slightly edited):** Let $n = 2013$. We can then write

$$\begin{aligned}
&\frac{(n+1)^4 + 4n^4}{n^2 + (2n+1)^2} - \frac{(n-1)^4 + 4n^4}{n^2 + (2n-1)^2} \\
=&\frac{[(n+1)^4 + 4n^4][n^2 + (2n-1)^2] - [(n-1)^4 + 4n^4][n^2 + (2n+1)^2]}{[n^2 + (2n+1)^2][n^2 + (2n-1)^2]}.
\end{aligned}$$

We then focus on expanding out the numerator. The first part of this is equal to

$$n^2(n+1)^4 + 4n^6 + 4n^4(2n-1)^2 + (n+1)^4(2n-1)^2.$$

Expanding the brackets and grouping like terms, this is equal to

$$25n^6 + 19n^4 - 5n^2 + 1.$$

On the other hand, expanding out the second pair of brackets gives us

$$n^2(n-1)^4 + 4n^6 + 4n^4(2n+1)^2 + (n-1)^4(2n+1)^2.$$

Once again, we can expand out the brackets further and group like terms, to yield

$$25n^6 + 19n^4 - 5n^2 + 1.$$

So the above expression is equal to

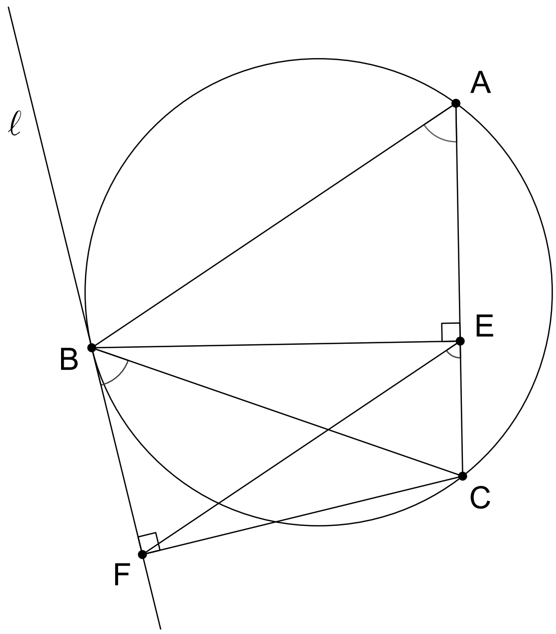
$$\frac{[25n^6 + 19n^4 - 5n^2 + 1] - [25n^6 + 19n^4 - 5n^2 + 1]}{[n^2 + (2n+1)^2][n^2 + (2n-1)^2]}$$

which is equal to 0.

**Problem 2** (Proposed by David Monk)

*In acute-angled triangle $ABC$, the foot of the perpendicular from $B$ to $AC$ is $E$. Let $\ell$ be the tangent to the circle $ABC$ at $B$. The foot of the perpendicular from $C$ to $\ell$ is $F$. Prove that $EF$ is parallel to $AB$.*

Most successful solutions to this problem involved the alternate segment theorem together with some results about angles in a cyclic quadrilateral.




**Solution by Balaji Krishna, Stanwell School:** First notice that $BECF$ is a cyclic quadrilateral since $\angle CEB + \angle BFC = 180°$. Therefore, $\angle CEF = \angle CBF$ by the theorem of angles in the same segment.

Also, by the alternate segment theorem, $\angle CAB = \angle CBF$. Thus $\angle CAB = \angle CEF$.

So by the converse of the corresponding angles theorem, lines $EF$ and $AB$ are parallel.

**Problem 3** (Proposed by Jeremy King) *A number written in base* 10 *is a string of* $3^{2013}$ *digit* 3*s. No other digit appears. Find the highest power of* 3 *which divides this number.*

Most solutions to this problem involved a method similar to the first solution presented; recursively breaking the number down into a product of many terms, each divisible by 3 but not by 9. The second solution is an unusual but neat alternative.

**Solution 1 by Frank Han, Dulwich College:** Let the number in question be $A$. Notice that $\frac{1}{3}A$ is a string of $3^{2013}$ 1s. Consider more generally the number $B_n$ which consists of a string of $3^n$ digit 1s.

Let $M_n$ be the number formed of a digit 1, $(3^n - 1)$ consecutive digits 0, another digit 1, another $(3^n - 1)$ consecutive digits 0 and then another 1. Notice that $B_n \cdot M_n = B_{n+1}$.

Since the digital sum of $M_n$ is 3, it is divisible by 3. However, since this is not divisible by 9, $M_n$ is not divisible by 9. So $B_{n+1}$ is divisible by exactly one higher power of 3 than $B_n$.

Now $B_1 = 111$ is divisible by $3^1$ but not by $3^2$, and so $B_n$ is divisible by $3^n$ but not by $3^{n+1}$. In particular, this means that $\frac{1}{3}A$ is divisible by $3^{2013}$ but not by $3^{2014}$. Hence $A$ is divisible by $3^{2014}$, but by no higher power of 3.

**Solution 2 by Kasia Warburton, Reigate Grammar School:** Call the number in question $N$. Then

$$\begin{aligned}
3N + 1 &= 10^{3^{2013}} \\
&= (3^2 + 1)^{3^{2013}} \\
&= (3^2)^{3^{2013}} + \binom{3^{2013}}{1}(3^2)^{3^{2013}-1} + \cdots \\
&\quad + \binom{3^{2013}}{3^{2013}-2}(3^2)^2 + \binom{3^{2013}}{3^{2013}-1}3^2 + 1
\end{aligned}$$

and so

$$\begin{aligned}
3N = (3^2)^{3^{2013}} &+ \binom{3^{2013}}{1}(3^2)^{3^{2013}-1} + \cdots \\
&+ \binom{3^{2013}}{3^{2013}-2}(3^2)^2 + \binom{3^{2013}}{3^{2013}-1}3^2.
\end{aligned}$$

Now consider the individual terms

$$\begin{aligned}
A_n &= \binom{3^{2013}}{3^{2013}-n}(3^2)^n \\
&= \frac{(3^{2013})(3^{2013}-1)\cdots(3^{2013}-n+1)}{n(n-1)(n-2)\cdots 1} \cdot 3^{2n} \\
&= \frac{(3^{2013}-1)\cdots(3^{2013}-n+1)}{(n-1)(n-2)\cdots 1} \cdot \frac{3^n}{n} \cdot 3^n \cdot 3^{2013} \\
&= \frac{(3^{2013}-1)}{1} \cdot \frac{(3^{2013}-2)}{2} \cdots \frac{(3^{2013}-(n-1))}{(n-1)} \cdot \frac{3^n}{n} \cdot 3^n \cdot 3^{2013}.
\end{aligned}$$

We will investigate which powers of 3 divide $A_n$.

For each $k < n$, if $3^{2013} - k$ is divisible by at least as many powers of 3 as $k$ is. Therefore none of the terms $\frac{3^{2013}-k}{k}$ have any powers of 3 in the denominator. So $A_n$ is divisible by at least as great a power of 3 as $\frac{3^n}{n} \cdot 3^n \cdot 3^{2013}$ is. However, $3^n$ is divisible by at least as great a power of 3 as $n$ is, and so $A_n$ is divisible by at least as great a power of 3 as $3^{2013+n}$.

We can check that $A_1 = 3^{2015}$ and that $3^{2016}|A_2$. By the above result, $3^{2016}|A_n$ for all $n > 2$, and so

$$3N = A_1 + A_2 + \cdots + A_{3^{2013}}$$

is divisible by $3^{2015}$, but not by $3^{2016}$. So the highest power of 3 that divides $N$ is $3^{2014}$.

**Problem 4** (Proposed by Jeremy King) *Isaac is planning a nine-day holiday. Every day he will go surfing, or water skiiing, or he will rest. On any given day he does just one of these three things. He never does different water-sports on consecutive days. How many schedules are possible for this holiday?*

There are a variety of ways of approaching this problem. Amongst successful solutions, one of the most common was to set up a recurrence, calculating the numbers of ways that Isaac could spend the first $n$ days of his holiday. A second approach involved splitting the holiday up into blocks of three days, and counting the number of ways that these blocks could be pieced together.

**Solution 1 by the setters:** Denote by $f(n)$ the number of possible holidays that conform to Isaac's rules, and last $n$ days. We are asked to find $f(9)$.

If he rests on the $n$th day, then he can surf, ski or rest on the $(n+1)$th day; three choices. If he waterskiis on the $n$th day, then he can rest or waterskii on the $(n+1)$th day; two choices. Similarly if he surfs, then he can either surf or rest. So the number of possible holidays of length $n+1$ is equal to three times the number of holidays of length $n$ that end with a rest day, plus two times the number of holidays of length $n$ that end with surfing, plus two times the number of holidays of length $n$ that end with waterskiing.

Thus the number of holidays over $(n+1)$ days is equal to twice the number of holidays over $n$ days, plus the number of holidays of $n$ days that end with a rest day. However, the number of holidays with $n$ days that end with a rest is equal to the number of holidays of length $(n-1)$. That is to say, that

$$f(n+1) = 2f(n) + f(n-1).$$

We can manually find that $f(1) = 3$ and $f(2) = 7$. Thereafter, we can calculate $f(3) = 17$, $f(4) = 41$, $f(5) = 99$, $f(6) = 239$, $f(7) = 577$, $f(8) = 1393$ and $f(9) = 3363$.

So the answer is $3363$.

**Solution 2 by Alexander Ma, Harrow School:** Split the 9 days into three sets of 3. Writing 'R' for resting, 'S' for surfing and 'W' for waterskiing, there are 17 possible schedules for a three day stretch, namely

SSS, SSR, SRR, SRS, SRW,
WWW, WWR, WRR, WRW, WRS,
RRR, RRS, RSR, RSS, RRW, RWR, RWW

How many ways can we piece three of these together? If we specify the last day of each of the first two sets of three, then we can calculate the number of possible ways - the restrictions are that if the first set ends in an S, then the second set must begin S or R, and if it ends in W, then the second set must begin W or R.

First set ends in S, second in S: $5 \times 4 \times 12 = 240$.
First set ends in S, second in R: $5 \times 5 \times 17 = 425$.
First set ends in S, second in W: $5 \times 3 \times 12 = 180$.
First set ends in R, second in S: $7 \times 5 \times 12 = 420$.
First set ends in R, second in R: $7 \times 7 \times 17 = 833$.
First set ends in R, second in W: $7 \times 5 \times 12 = 420$.
First set ends in W, second in S: $5 \times 3 \times 12 = 180$.
First set ends in W, second in R: $5 \times 5 \times 17 = 425$.
First set ends in W, second in W: $5 \times 4 \times 12 = 420$.

The total number of possible schedules is this the sum of these nine numbers,

$$240 + 425 + 180 + 420 + 833 + 420 + 180 + 425 + 420 = 3363.$$

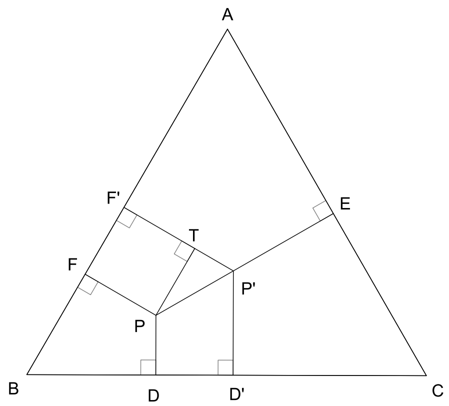
**Problem 5** (Proposed by Karthik Tadinada) *Let $ABC$ be an equilateral triangle, and $P$ be a point inside the triangle. Let $D, E$ and $F$ be the feet of the perpendiculars from $P$ to the sides $BC, CA$ and $AB$ respectively. Prove that*

*a)* $AF + BD + CE = AE + BF + CD$, *and*

*b)* $[APF] + [BPD] + [CPE] = [APE] + [BPF] + [CPD]$.

The most common approach to this problem was to observe that the required equalities hold when $P$ is some natural point such as the centre of the triangle, and then show that as $P$ moves around, the two sides are unaffected. We also present a very different, clean approach that avoids any need for calculation.

**Solution 1 by Tian Bei Li, Concord College:** If $P$ is the centre of triangle $ABC$, then the result holds by symmetry. We can reach an arbitrary point $P$ inside $ABC$ by starting at the centre $P_0$ and moving along the line $PD$ and then along the line $PE$ (for some distance in either direction). We will prove that the equality a) holds by proving that it is preserved by such movements of $P$.




Consider points $P$ and $P'$ such that $P'$ lies on the segment $PE$, as shown in the above diagram. Let the feet of the perpendiculars from $P'$ to $BC$ and $AB$ be $D'$ and $F'$ respectively. Also let $T$ be the foot of the perpendicular from $P$ to $P'F'$.

By considering quadrilateral $P'EAF'$, $\angle TPP' = \angle EAF' = 60°$. So $PT = PP' \sin 60°$. But $PTF'F$ is a rectangle and so $FF' = PP' \sin 60°$. But by symmetry, $DD' = PP' \sin 60° = FF'$.

Thus

$$\begin{aligned} AF' + BD' + CE &= AF - FF' + BD + DD' + CE \\ &= AF + BD + CE \end{aligned}$$

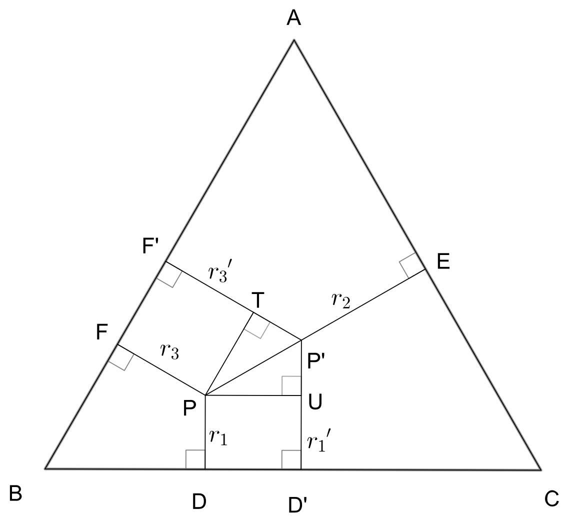
and

$$\begin{aligned} AE + BF' + CD' &= AE + BF' + FF' + CD - DD' \\ &= AE + BF + CD \end{aligned}$$

and so if $AF + BD + CE = AE + BF + CD$ then also $AF' + BD' + CE = AE + BF' + CD'$.

This equality is also preserved when moving along the line $PE$ *away* from $E$, by considering the same situation with $P$ and $P'$ reversed. Similarly it is preserved when moving along the line $PD$. Since it holds when $P$ is the centroid of $ABC$, it holds for all points $P$ inside $ABC$.

To prove part b), we'll use the same method; the result is clearly true when $P$ is the centroid of the triangle, and we will show that the equality is preserved by movements along the lines $PD$ and $PE$.



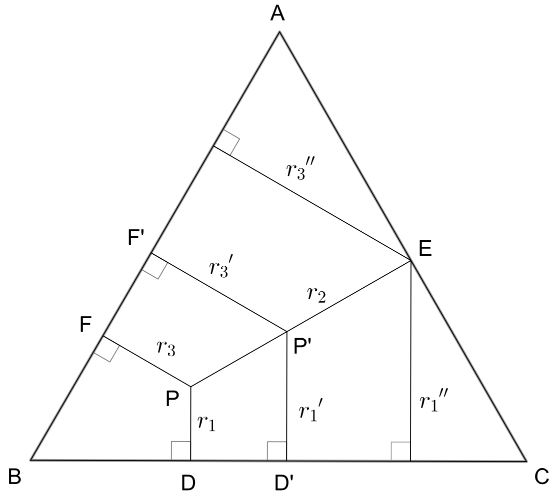

For ease of notation, we'll call the lengths $PD = r_1, P'D' = r_1', PE = r_2$ and so on as indicated in the diagram. Also, let $PP' = d$. First of all note that, as $\angle PP'T = 60°$, $P'T = P'U = \frac{d}{2}$ and so $r_1 - r_1' = \frac{d}{2} = r_3 - r_3'$. (Note that it follows from this that $r_1 + r_2 + r_3 = r_1' + r_2' + r_3'$, but we shall not actually use this fact.)

By repeatedly using the formula 'area $= \frac{1}{2}$base $\times$ height',

$$\begin{aligned} &[AP'F'] + [BP'D'] + [CP'E] - [APF] - [BPD] - [CPE] \\ &= (r_3'AF' + r_1'BD' + r_2CE - r_3AF - r_1BD - r_2CE) \\ &= (r_3 + \tfrac{1}{2}d)(AF - \tfrac{d}{2}\cos 30°) - r_3AF \\ &+ (r_1 + \tfrac{1}{2}d)(BD - \tfrac{d}{2}\cos 30°) - r_1BD \\ &+ (r_2 - d)CE - r_2CE \\ &= \tfrac{d}{2}(AF + BD - 2CE + \sqrt{3}(r_1 - r_3)). \qquad (1) \end{aligned}$$

We'll aim to unpick this last term, $r_1 - r_3$. From our earlier work, $r_1 - r_3 = r_1' - r_3'$. This holds for all $P'$ along the line $PE$, and so we could pick $P'' = E$.

Thus, when $r_1''$ is the perpendicular distance from $E$ to $BC$ and $r_3''$ is the perpendicular distance from $E$ to $AB$, we have that $(r_1 - r_3) = (r_1'' - r_3'')$.




Now since $\angle D'CE = 60°$, we know that $r_1'' = CE \sin 60°$ and similarly $r_3'' = EA \sin 60°$. So

$$(r_1 - r_3) = \frac{\sqrt{3}}{2}(CE - (AC - CE)) = \sqrt{3}CE - \frac{\sqrt{3}}{2}AC.$$

We know from part a) that $AF + BD = \frac{3}{2}AC - CE$, and so we have that (1) is equal to

$$\tfrac{d}{2}(\tfrac{3}{2}AC - 3CE + 3CE - \tfrac{3}{2}AC) = 0.$$

Thus

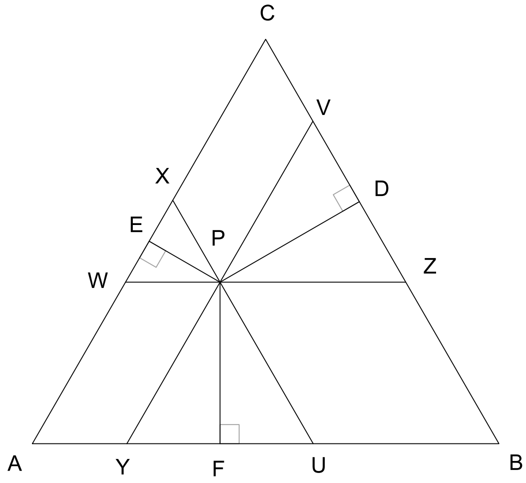
$$[AP'F'] + [BP'D'] + [CP'E] = [APF] + [BPD] + [CPE]$$

and so

$$[AP'E] + [BP'F] + [CP'D] = [APE] + [BPF] + [CPD].$$

Combining these two, we see that if the desired equality holds for $P$, then it also does for $P'$. As before, it follows then that it holds for any choice $P'$ on the line $PE$, and then for any $P'$ inside $ABC$ by moving along the line $AE$ and then the line $AD$.

**Solution 2 by the setters:** Start off by drawing three lines through $P$, parallel to each of the three sides; $UX$ parallel to $BC$, $VY$ parallel to $CA$ and $WZ$ parallel to $AB$. (The six points $U, V, W, X, Y$ and $Z$ are on the sides of the triangle as shown in the diagram.)




Since $ABC$ is equilateral, each of these three lines meets the sides of the triangle at a $60^\circ$ angle. So the triangles $PZV$, $PXW$ and $PYU$ are all equilateral. As a result, the perpendicular $PD$ cuts the line segment $ZV$ in half, and similarly for $PE$ and $PF$. We thus have that

$$VD = DZ, \tag{2}$$
$$WE = EX, \text{ and} \tag{3}$$
$$UF = FY. \tag{4}$$

However we also know that $BUXC$ is an isoceles trapezium. So $BU$ and $XC$ are the same length. Applying the same logic to the trapezia $CVYA$ and $AWZB$ gives us that

$$BU = XC, \tag{5}$$
$$CV = YA, \text{ and} \tag{6}$$
$$AW = ZB. \tag{7}$$

We can divide up each of the six lengths in question as follows:

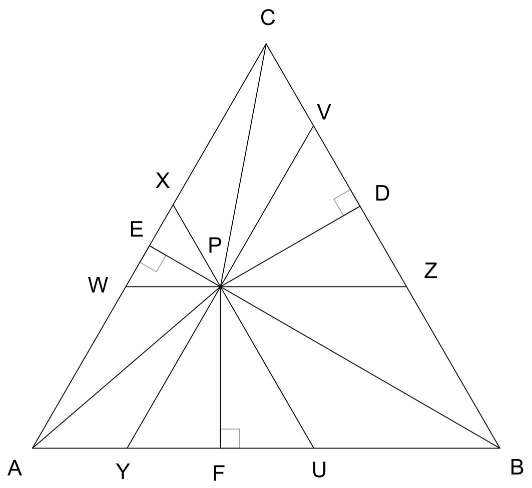
$$AF = AY + YF$$
$$BD = BZ + ZD$$
$$CE = CX + XE$$
$$AE = AW + WE$$
$$BF = BU + UF$$
$$CD = CV + VD$$

Thus the combination of (2), (3), (4), (5), (6) and (7) tells us exactly that

$$AF + BD + CE = AE + BF + CD,$$

as the question required.

For part b), we'll add the lines $AP$, $BP$ and $CP$ to the diagram.




The quadrilateral $AYPW$ is a parallelogram. Therefore, the two triangles $PAW$ and $APY$ are congruent. Similarly, the triangles $PBU$ and $BPZ$ are congruent, and the triangles $PCV$ and $CPX$ are congruent.

Exactly as in part a), we know that the line $PD$ splits the triangle $PVZ$ in half, and so triangles $PDV$ and $PDZ$ are congruent. Similarly, triangles $PEW$ and $PEX$ are congruent, and triangles $PFU$ and $PFY$ are congruent.

We have learnt that six pairs of triangles are congruent, and so each pair has equal area:

$$[PAW] = [APY], \tag{8}$$
$$[PBU] = [BPZ], \tag{9}$$
$$[PCV] = [CPX], \tag{10}$$
$$[PDV] = [PDZ], \tag{11}$$
$$[PEW] = [PEX], \text{ and} \tag{12}$$
$$[PFU] = [PFY]. \tag{13}$$

Summing up the left and right sides of (8), (9), (10), (11), (12) and (13) tells us exactly that

$$[APF] + [BPD] + [CPE] = [APE] + [BPF] + [CPD],$$

as required.

**Problem 6** (Proposed by Graham Hoare) *The angles $A, B$ and $C$ of a triangle are measured in degrees, and the lengths of the opposite sides are $a, b$ and $c$ respectively. Prove that*

$$60 \leq \frac{aA + bB + cC}{a + b + c} \leq 90$$

The two parts of the this inequality can be tackled separately. Most successful solutions used the rearrangement inequality for the left side, and the triangle inequality for the right side, as the solution presented below does.

**Solution by Freddie Illingworth, Magdalen College School:** Without loss of generality we may assume that $a \geq b \geq c$. Since the largest angle is opposite the longest side in any triangle, and the smallest angle is opposite the shortest side, we then also have that $A \geq B \geq C$.

That is to say, that $a, b, c$ and $A, B, C$ are both non-increasing sequences. Then it follows from the rearrangement inequality that

$$\begin{aligned} aA + bB + cC &\geq aB + bC + cA \\ aA + bB + cC &\geq aC + bA + cB. \end{aligned}$$

Summing these two inequalities, and adding $aA + bB + cC$ to each side gives

$$\begin{aligned} 3(aA + bB + cC) &\geq aA + aB + aC \\ &\quad +bA + bB + bC \\ &\quad +cA + cB + cC \\ &= (a + b + c)(A + B + C) \\ &= 180(a + b + c). \end{aligned}$$

since the angles in a triangle sum to $180°$. Dividing through by $3(a + b + c)$ gives us

$$\frac{aA + bB + cC}{a + b + c} \geq 60.$$

Now we prove the other inequality. We continue to assume that $a \geq b \geq c$ and consequently that $A \geq B \geq C$. The triangle inequality states that $a < b + c$, and so

$$aA < (b + c)A = bA + cA.$$

We also know that $bB \leq aB$ and $cC \leq aC$. Thus

$$\begin{aligned} aA + bB + cC &< aB + aC + bA + cA \\ &< aB + aC + bA + bC + cA + cB \end{aligned}$$

Thus

$$\begin{aligned}
2(aA + bB + cC) &< aA + aB + aC \\
&\quad + bA + bB + bC \\
&\quad + cA + cB + cC \\
&= (a + b + c)(A + B + C) \\
&= 180(a + b + c).
\end{aligned}$$

Dividing through by $2(a + b + c)$ gives us that

$$\frac{aA + bB + cC}{a + b + c} < 90$$

as required.

# Solutions
# 2014

## BMO Round 1

**Problem 1** (Proposed by Julian Gilbey) *Place the following numbers in increasing order of size, and justify your reasoning:*

$$3^{3^4}, 3^{4^3}, 3^{4^4}, 4^{3^3} \textit{ and } 4^{3^4}$$

*Note that $a^{b^c}$ means $a^{(b^c)}$*

**Solution 1 by Harry Metrebian, Winchester College:** The five numbers are $3^{81}, 3^{64}, 3^{256}, 4^{27}$ and $4^{81}$. Clearly

$$3^{64} < 3^{81} < 3^{256}.$$

Now $4^{27} = (4^3)^9 = 64^9$ and $3^{64} = (3^4)^{16} = 81^{16}$. We can see that $64^9 < 81^{16}$ and so

$$4^{27} < 3^{64} < 3^{81} < 3^{256}.$$

Since $4^{81} > 3^{81}$ it remains to compare $4^{81}$ and $3^{256}$. However $4^{81} = (4^3)^{27} = 64^{27}$ and $3^{256} = (3^4)^{64} = 81^{64}$ and it is clear that $64^{27} < 81^{64}$. So

$$4^{27} < 3^{64} < 3^{81} < 4^{81} < 3^{256},$$

that is,

$$4^{3^3} < 3^{4^3} < 3^{3^4} < 4^{3^4} < 3^{4^4}.$$

**Solution 2 by Bryan Ng, Bedford School:** By calculation, $3^{3^4} = 3^{81}$, $3^{4^3} = 3^{64}$, $3^{4^4} = 3^{256}$, $4^{3^3} = 4^{27}$ and $4^{3^4} = 4^{81}$. We can take the tenth root of all five numbers without affecting the ordering, and since this is equivalent to dividing the exponents by ten, it remains to sort the following numbers:

$$3^{8.1}, 3^{6.4}, 3^{25.6}, 4^{2.7} \text{ and } 4^{8.1}.$$

Then

$$\begin{aligned} 6561 = 3^8 < 3^{8.1} < 3^9 = 19683, \\ 729 = 3^6 < 3^{6.4} < 3^7 = 2187, \\ 16 = 4^2 < 4^{2.7} < 4^3 = 64, \\ 65536 = 4^8 < 4^{8.1} < 4^9 = 262144, \end{aligned}$$

and

$$262144 < 6561^3 = (3^8)^3 = 3^{24} < 3^{25.6}.$$

From these we can read off the order:

$$4^{2.7} < 3^{6.4} < 3^{8.1} < 4^{8.1} < 3^{25.6}$$

and so

$$4^{3^3} < 3^{4^3} < 3^{3^4} < 4^{3^4} < 3^{4^4}.$$

**Problem 2** (Proposed by Gerry Leversha) *Positive integers $p$, $a$ and $b$ satisfy the equation $p^2 + a^2 = b^2$. Prove that if $p$ is a prime greater than 3, then $a$ is a multiple of 12 and $2(p + a + 1)$ is a perfect square.*

**Solution 1 by Harry Metrebian, Winchester College:** Rearranging and factorizing, we have that

$$p^2 = b^2 - a^2 = (b + a)(b - a).$$

The only positive factors of $p^2$ are 1, $p$ and $p^2$. We cannot have that $b + a = b - a = p$ since $a$ is positive, so it must be the case that $b + a = p^2$ and $b - a = 1$. So

$$b = \frac{p^2 + 1}{2} \text{ and } a = \frac{p^2 - 1}{2}$$

and therefore

$$2a = p^2 - 1 = (p + 1)(p - 1).$$

Since $p$ is odd, both $p + 1$ and $p - 1$ are even, and moreover one of them is divisible by 4. Hence 8 divides $2a$. Since $p$ is not divisible by 3, one of $p + 1$ and $p - 1$ is divisible by 3 and so 3 divides $2a$. As 8 and 3 are coprime, it follows that 24 divides $2a$ and 12 divides $a$, as required.

Now

$$\begin{aligned} 2(p + a + 1) &= 2(p + \tfrac{p^2-1}{2} + 1) \\ &= 2p + p^2 - 1 + 2 \\ &= p^2 + 2p + 1 \\ &= (p + 1)^2 \end{aligned}$$

which is to say that $2(p + a + 1)$ is a perfect square, as required.

**Solution 2 by James Roper, Upton Court Grammar School (slightly edited):** Considering the equation modulo 3, and using the fact that $b^2$ is congruent to either 0 or 1 modulo 3, we see that at least one of $p^2$ and $a^2$ is congruent to 0 modulo 3. It can't be $p^2$, and therefore $a^2 \equiv 0 \mod 3$ and $p^2 \equiv b^2 \equiv 1 \mod 3$. In particular, 3 divides $a$.

Considering the equation modulo 4, and using the fact that $b^2$ is congruent to either 0 or 1 modulo 4, we see that at least one of $p^2$ and $a^2$ is congruent to 0 modulo 4. It can't be $p^2$, and therefore $a^2 \equiv 0 \mod 4$, and so $a$ is even but $b$ and $p$ are both odd.

Factorizing the original equation, $a^2 = (b - p)(b + p)$. Since both $b$ and $p$ are odd, one of $(b-p)$ and $(b+p)$ must be divisible by 4, and since the other is also even, it follows that $8$ divides $a^2$. Therefore $4$ divides $a$. We have shown that $a$ is divisible by both $3$ and $4$, and since $3$ and $4$ are coprime, it follows that $a$ is divisible by $12$.

Factorizing the original equation again, $p^2 = (b-a)(b+a)$. The only positive factors of $p^2$ are $1$, $p$ and $p^2$. We cannot have that $b+a = b-a = p$ since $a$ is positive, so it must be the case that $b+a = p^2$ and $b-a = 1$. So $b+a = p^2$ and $b-a = 1$. Thus $p+a+1 = b+p$.

Let's consider all primes $q$ that divide $b+p$. If $q$ is not $2$ or $p$, $q$ cannot divide $b-p$ (as then it would divide the difference $(b+p)-(b-p) = 2p$) and so $q$ divides $b+p$ exactly as many times as it divides $a^2$, which is an even number of times. It cannot be the case that $p$ divides $b+p$, as then it would divide both $b$ and $a$, but $b-a = 1$ and so $p$ would divide $1$. We know that exactly one of $(b+p)$ and $(b-p)$ is congruent to $2$ modulo $4$, and since $2$ must divide $a^2$ an even number of times, $2$ divides each of $b+p$ and $b-p$ an odd number of times. So $2$ divides $2(b+p)$ an even number of times.

Therefore every prime that divides $2(b+p) = 2(p+a+1)$ divides it an even number of times. Therefore $2(p+a+1)$ is a perfect square.

**Problem 3** (Proposed by Daniel Griller) *A hotel has ten rooms along each side of a corridor. An olympiad team leader wishes to book seven rooms on the corridor so that no two reserved rooms on the same side of the corridor are adjacent. In how many ways can this be done?*

**Solution by Agnijo Banerjee, Grove Academy:** Number the rooms on each side of the corridor 1 to 10. We'll demonstrate a way in which you can pick $n$ non-adjacent rooms on one side of the corridor.

Pick integers $a_1, a_2, \dots a_n$ such that $1 \leq a_1 < a_2 < \cdots < a_n \leq 11 - n$. Then pick rooms $a_1, a_2 + 1, a_3 + 2, \dots a_n + n - 1$. We can see that these room numbers are increasing, and since $a_{i+1} > a_i$, $(a_{i+1} + i) - (a_i + i - 1) \geq 2$. So no two such rooms are adjacent.

Conversely, suppose that we have an arrangement of $n$ non-adjacent rooms on one side of the corridor. By ordering them in increasing (number) order, and subtracting $(i - 1)$ from the number of the $i$th room, we get a sequence $b_1, b_2, \dots b_n$ with $1 \leq b_1 < b_2 < \cdots < b_n \leq 11 - n$, so all possible arrangements of $n$ non-adjacent rooms arise in the way described above. There are $\binom{11-n}{n}$ ways to pick $a_1, \dots a_n$ and so $\binom{11-n}{n}$ ways to pick $n$ non-adjacent rooms on one side of the corridor. (Note that it is not possible to pick six or more non-adjacent rooms on one side of the corridor).

If we label the two sides of the corridor $A$ and $B$, then we have four choices: picking 5 rooms from $A$ and 2 from $B$, picking 4 rooms from $A$ and 3 from $B$, picking 3 rooms from $A$ and 4 rooms from $B$, or picking 2 rooms from $A$ and 5 from $B$. Since what we do on one side of the corridor does not affect the other side, there are

$$\begin{aligned}&\binom{6}{5} \cdot \binom{9}{2} + \binom{7}{4} \cdot \binom{8}{3} + \binom{8}{3} \cdot \binom{7}{4} + \binom{9}{2} \cdot \binom{6}{5} \\ &= 216 + 1960 + 1960 + 216 \\ &= 4352\end{aligned}$$

ways of assigning the rooms.

**Problem 4** (Proposed by Jeremy King) *Let $x$ be a real number such that $t = x + x^{-1}$ is an integer greater than 2. Prove that $t_n = x^n + x^{-n}$ is an integer for all positive integers $n$. Determine the values of $n$ for which $t$ divides $t_n$.*

**Solution by Harvey Yau, Ysgol Dyffryn Taf:** Firstly, $t_1 = t$ which is an integer. Then $t_2 = x^2 + 2 + x^{-2} - 2 = t^2 - 2$ which is also an integer. More generally,

$$\begin{aligned} t_k &= x^k + x^{-k} \\ &= x^k + x^{k-2} + x^{2-k} + x^{-k} - x^{k-2} - x^{2-k} \\ &= (x + x^{-1}) \cdot (x^{k-1} + x^{1-k}) - (x^{k-2} + x^{2-k}) \\ &= t \cdot t_{k-1} - t_{k-2} \end{aligned}$$

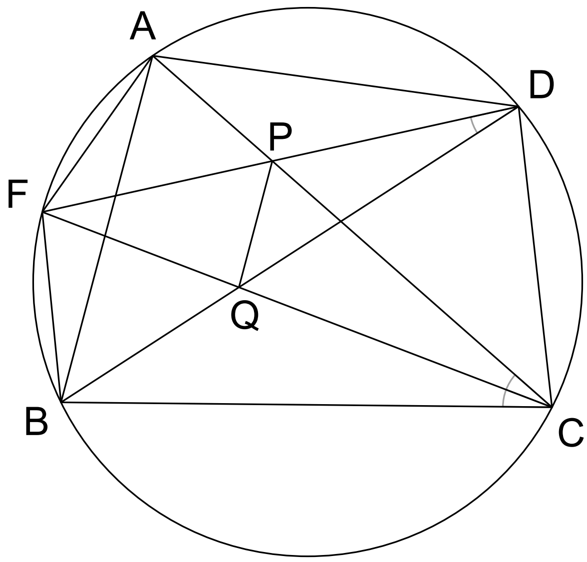
Therefore if $t_{k-1}$ and $t_{k-2}$ are both integers, $t_k$ is also an integer. We have shown that $t_1$ and $t_2$ are integers, and so it follows by induction that $t_n$ is an integer for all $n$.

We claim that $t$ divides $t_n$ if and only if $n$ is odd.

Since $t > 2$, we can see that $t$ does not divide $t_2$. Suppose that $k$ is odd, and $t$ divides $t_{k-2}$ but not $t_{k-1}$. Then, as $t_k = t \cdot t_{k-1} - t_{k-2}$ it follows that $t$ divides $t_k$. On the other hand, suppose that $k$ is even, and $t$ divides $t_{k-1}$ but not $t_{k-2}$. Again, $t_k = t \cdot t_{k-1} - t_{k-2}$, from which it follows that $t$ does not divide $t_k$. So our claim is true by induction.

**Problem 5** (Proposed by David Monk) *Let $ABCD$ be a cyclic quadrilateral. Let $F$ be the midpoint of the arc $AB$ of its circumcircle that does not contain $C$ or $D$. Let the lines $DF$ and $AC$ meet at $P$ and the lines $CF$ and $BD$ meet at $Q$. Prove that the lines $PQ$ and $AB$ are parallel.*

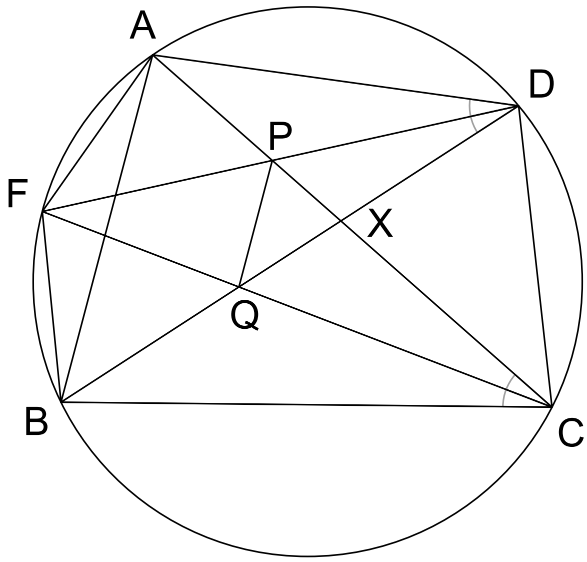
**Solution 1 by Sooming Jang, Badminton School:** By the theorem of angles in the same segment, $\angle BCF = \angle BDF$. But since the chord $FA$ has the same length as the chord $BF$, the angle subtended by the chord $FA$ is equal to the angle subtended by the chord $BF$. So $\angle FCA = \angle BCF$.




As $\angle QDP = \angle BDF = \angle BCF = \angle FCA = \angle QCP$, it follows by the converse of the theorem of angles in the same segment that quadrilateral $PQCD$ is cyclic.

Then $\angle CPQ = \angle CDQ$ by the theorem of angles in the same segment, and $\angle CDQ = \angle CDB = \angle CAB$ by the same theorem. So $\angle CPQ = \angle CAB$. From this it follows, by the converse of the theorem of corresponding angles, that lines $PQ$ and $AB$ are parallel.

**Solution 2 by Harry Metrebian, Winchester College:** Let $X$ be the point of intersection of $AC$ and $BD$. Since chords $FA$ and $BF$ are of equal length, $\angle FDA = \angle BDF = \angle FCA = \angle BCF$ by the theorem of angles in the same segment. We have then that $\angle XDA = \angle BCX$ and $\angle AXD = \angle BXC$ (vertically opposite angles). Therefore triangle $AXD$ is similar to triangle $BXC$.




So $\frac{XC}{BC} = \frac{XD}{AD}$. But also, by the angle bisector theorem in triangles $BXC$ and $AXD$ respectively, $\frac{XQ}{BQ} = \frac{XC}{BC}$ and $\frac{XP}{PA} = \frac{XD}{AD}$. Combining these two, $\frac{XQ}{BQ} = \frac{XP}{AP}$. Then $\frac{BQ}{XQ} = \frac{PA}{XP}$ and by adding $1 = \frac{XQ}{XQ} = \frac{XP}{XP}$ to both sides we have that $\frac{XB}{XQ} = \frac{XA}{XP}$.

Since $\angle BXA = \angle QXP$, it follows that triangles $BXA$ and $QXP$ are similar. So $\angle XPQ = \angle XAB$ and so the lines $AB$ and $PQ$ are parallel by the converse of the theorem of corresponding angles.

**Problem 6** (Proposed by Julian Gilbey) *Determine all functions $f(n)$ from the positive integers to the positive integers which satisfy the following condition: whenever $a$, $b$ and $c$ are positive integers such that $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$, then*

$$\frac{1}{f(a)}+\frac{1}{f(b)}=\frac{1}{f(c)}.$$

**Solution by Chikashi Rison, The Leys School:** We claim that $f(mn) = mf(n)$ for all positive integers $m$ and $n$. We aim to prove this by induction on $m$. It is clearly true (for all $n$) when $m = 1$.

Suppose that this is true (for all $n$) when $m = k$. For some arbitrary $n$, let $a = (k+1)n$, $b = k(k+1)n$ and $c = kn$. Note that, by factorizing, we first have that

$$\begin{aligned}\frac{1}{a}+\frac{1}{b} &= \frac{1}{(k+1)n}+\frac{1}{k(k+1)n} \\ &= \frac{1}{n}\left(\frac{1}{k+1}+\frac{1}{k(k+1)}\right)\end{aligned}$$

and then since $\frac{1}{k}=\frac{1}{k+1}+\frac{1}{k(k+1)}$, it follows that

$$\begin{aligned}\frac{1}{a}+\frac{1}{b} &= \frac{1}{n}\cdot\frac{1}{k} \\ &= \frac{1}{c}\end{aligned}$$

and so we can use the condition. Then

$$\begin{aligned}\frac{1}{f(a)}+\frac{1}{f(b)} &= \frac{1}{f(c)} \\ \frac{1}{f((k+1)n)}+\frac{1}{f(k(k+1)n)} &= \frac{1}{f(kn)} \\ \frac{1}{f((k+1)n)}+\frac{1}{kf((k+1)n)} &= \frac{1}{kf(n)}\end{aligned}$$

where we use the induction hypothesis to go from the second line to the third. But, factorizing,

$$\begin{aligned}\frac{k+1}{k}\cdot\frac{1}{f((k+1)n)} &= \frac{1}{k}\cdot\frac{1}{f(n)} \\ \frac{1}{f((k+1)n)} &= \frac{1}{(k+1)f(n)}\end{aligned}$$

and so $f((k+1)n) = (k+1)f(n)$, completing the induction step.

So we have that $f(mn) = mf(n)$ for all positive integers $m$ and $n$. In particular, $f(m) = mf(1)$ and so any functions $f$ that satisfy this condition are of the form $f(n) = cn$ for some positive integer constant $c$. It is clear that all such functions do indeed satisfy the condition.