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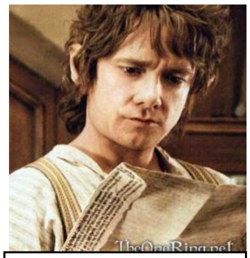

What have I got in my pocket? Your C2 success, that's what.

The Hobbitey Guide to C2 (Hobbit Edition)

1. Algebraic Division and the Factor & Remainder Theorems

General pointers:

- **Bilbo Tip 1:** Be careful when subtracting negative numbers (i.e. you add!). e.g.

$$\begin{array}{r} 2x^2 + 3x \\ \underline{-(2x^2 - 3x)} \\ 6x \end{array}$$

- **Bilbo Tip 2:** Be sure that when you write out your polynomial, you use consecutive powers.
  e.g. $9x^3 - 3 \quad = \quad 9x^3 + 0x^2 + 0x - 3$
  The long-division method won't work otherwise.

Harder Example:

Divide $8x^3 - x + 3$ by $2x^2 - x$

Bilbo Tip 2 is relevant here.

Bilbo Tip 1 is relevant here.

```
                 4x   +  2
          ________________________
2x^2 - x  |  8x^3 + 0x^2 - x + 3
             8x^3 - 4x^2
             -----------
                    4x^2 - x
                    4x^2 - 2x
                    ---------
                           x + 3
```

So we get $4x + 2$ with a remainder of $x + 3$.

In this example:

- $8x^3 - x + 3$ was the **dividend**: i.e. the thing we're dividing.
- $2x^2 - x$ was the **divisor**: i.e. what we're dividing by.
- $4x + 2$ was the **quotient**: i.e. the whole number of times the divisor goes into the dividend.
- $x + 3$ was the **remainder**.

Remainder theorem: To find the remainder when we divide $f(x)$ by $(ax - b)$, we can just evaluate $f(\frac{b}{a})$.

Bilbo Tip 3: To remember this more easily, thing about what value of x would make the divisor 0. e.g. If you were dividing by $(x - 4)$, then $x = 4$ would make this bracket 0, so evaluate $f(4)$ to get the remainder.

Example: "Find the remainder when we divide $x^3 - x$ by $(2x + 3)$."

$$f\left(-\frac{3}{2}\right) = \left(-\frac{3}{2}\right)^3 - \left(-\frac{3}{2}\right) = -\frac{15}{8}$$

Bilbo Tip 4: Be very careful about substituting your negatives in. Remember that a negative cubed is negative, and a negative squared is positive.

Factor Theorem: If we find (using the Remainder Theorem) that the remainder is 0, then by definition, what we divided by (the divisor) must be a factor.

Bilbo Tip 5: We can use the Factor Theorem to factorise difficult polynomials (e.g. cubics and quartics).

Example: Factorise $f(x) = 2x^3 + 9x^2 - 6x - 5$.
'Guesstimate' a factor $(x-1)$. Evaluate $f(1)$ to check: $2 + 9 - 6 - 5 = 0$. Since the remainder is 0, $(x-1)$ is a factor. We need to find what the other factors are now. Now use algebraic division to find that we get $2x^2 + 11x + 5$ when we divide $2x^3 + 9x^2 - 6x - 5$ by $(x-1)$. We know from GCSE how to factorise the quadratic $2x^2 + 11x + 5$, which is $(x+5)(2x+1)$.
Thus $2x^3 + 9x^2 - 6x - 5 = (x-1)(x+5)(2x+1)$.

<u>2. Sine and Cosine Rule</u>

- When two angles are involved (e.g. one known, and one you want to try and find out), use the sine rule.
- Otherwise, use the cosine rule.
- For the cosine rule, suppose the unknown angle is at C. Then your formula to use would be $c^2 = a^2 + b^2 - 2ab\cos C$. Remember that an angle and the opposite side use the same letter.
- Area of triangle $= \frac{1}{a} ab \sin C$

<u>3. Exponentials and Logarithms</u>

Gandalf <u>Ridiculously Important</u> Tip: When solving equations involving logs, you're trying to gradually get towards a point where there's either (a) a log on one side of the equation on its own, e.g. $\boldsymbol{log_3(2x) = 4}$, or (b) a log on both sides with the same base, e.g. $\boldsymbol{log_3(x+1) = log_3(3x-2)}$.

Example: Solve $\boldsymbol{\log_2\left(\frac{2x+1}{x}\right) = 2}$.

You might be tempted to use your law of logs to expand the LHS to $\log_2(2x+1) - \log_2(x)$. But you need to think: does this actually help me? Now we would have two logs floating about, and it's hard to know how to proceed forward from there.

Instead, in case (a) where we already have a log isolated on one side of the equation, we should rearrange the equation to put it in exponential form. We can imagine the result of the log being inserted between the base and the argument, i.e.

$$\log_2\left(\frac{2x+1}{x}\right) = 2 \quad \Longrightarrow \quad 2^2 = \frac{2x+1}{x}$$

And then solve from there.

Example: $\boldsymbol{\log_2(3x) + \log_2(4x) = \log_2(24)}$

We could simplify the LHS to $\log_2(12x^2)$ using laws of logs. Then since $\log_2(12x^2) = \log_2(24)$, we can remove the log from both side to get $12x^2 = 24$. (Note: this is NOT because log is some kind of 'quantity' that we can divide both sides by: log is a function).

Gandalf Tip 2: Get your laws of logs right! I often see students do $\boldsymbol{\log(a+b) = \log(a) + \log(b)}$, which is not in general true. Make sure you revise and practice using these laws.

Gandalf Tip 3: If you ever have a variable or an expression involving a variable in a power, e.g. $\boldsymbol{3^{x+1} = 7}$, then your instinct should be to take logs of both sides (using the same base, so 3 in this example).

Gandalf Tip 4: Many questions will require you to form a quadratic equation which you can then solve.

Example: $2^{2x+1} - 5 \cdot 2^x - 3 = 0$

Your first step should be to turn 2^{2x+1} into an expression involving 2^x. To do this, just remember your laws of indices: $2^{2x+1} = 2^1 2^{2x} = 2(2^x)^2$. Using this, we have:

$$2(2^x)^2 - 5 \cdot 2^x - 3 = 0$$

This look like a quadratic equation! Substituting $y = 2^x$ gives us: $2y^2 - 5y - 3 = 0$. Solving, $(2y+1)(y-3) = 0$ so $y = -\frac{1}{2}$ or $y = 3$.

Substituting back, then $2^x = -\frac{1}{2}$ or $2^x = 3$. Using 'Tip 3', we can take logs of both sides to get $\log_2 2^x = \log_2\left(-\frac{1}{2}\right)$ and $\log_2 2^x = \log_2 3$. Then:

$$x = \log_2\left(-\frac{1}{2}\right) \; or \; x = \log_2(3)$$

We reject the first solution because we can't log a negative number. If we wanted to evaluate the second solution on the calculator (and don't have the nice button that lets us specify the base), we could change the base to 10 (since 'log' on a calculator on a calculator is by default base 10):

$x = \frac{\log_{10} 3}{\log_{10} 2} = 1.58 \; (to \; 3sf)$

Gandalf Tip 5: When you have some equation/expression involving logs with different bases, you first step should be to change the bases so that they're consistent. It's best the change the base to: (a) a constant and (b) the smallest value. So if for example you had log_2 and log_4 both floating around, then you want both to be log_2. And if you have log_3 and log_x, you want both in terms of log_3.

Example: Solve $\log_4 x + \log_2 x = 5$

Changing the log base 4 to log base 2 as per the tip:

$$\frac{\log_2 x}{\log_2 4} + \log_2 x = 5$$

$$\frac{1}{2}\log_2 x + \log_2 x$$

$$\log_2 x^{\frac{1}{2}} + \log_2 x = 5$$

$$\log_2 x^{\frac{3}{2}} = 5$$

$$2^5 = x^{\frac{3}{2}}$$

$$x = (2^5)^{\frac{2}{3}} = 2^{\frac{10}{3}}$$

Because $\log_2 4 = 2$

The reason I've moved the coefficient to the power is so that in the next step we can use a law of logs to combine the two logs into one.

We've isolated a single log on its own, so now we can rearrange!

Final Gandalf Tip: Avoid horrors like $\log(3x^2) \to 2\log(3x)$. Because of BIDMAS, $3x^2 = 3 \times x^2$ and is not $(3x)^2$. Thus $\log(3x^2) \to \log(3) + \log(x^2) \to \log(3) + 2\log(x)$

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4. Coordinate Geometry in the x-y plane

Smaug Tip 1: Exam question asking you to find the centre and/or radius of a circle for a given equation? No sweat! Just put it in the form $(x-a)^2+(y-b)^2=r^2$ by completing the square.

Example: Find the centre and radius of the circle given by the equation $x^2+y^2=6x+2y+6$

$$(x-3)^2-9+(y-1)^2-1=6$$
$$(x-3)^2+(y-1)^2=16$$

So the centre is $(3,1)$ and the radius is 4.

Smaug Tip 2: Exam question asking the coordinates for which a circle and a line intersect? Just arrange the equation of the line and substitute it into the circle equation, then solve.

Other scenarios:

Proving that a line doesn't intersect a circle.	Substitute equation of line into that of circle. Show that discriminant of resulting quadratic equation is negative.
Showing that a line is the diameter of a circle.	Find the radius and centre of the circle. Show that the line is double the radius of the circle, and the midpoint of the line is the centre of the circle.
Showing that a line is the tangent of a circle.	Substitute equation of line into that of circle. Show that discriminant of resulting quadratic equation is 0: indicating that the line and circle intersect exactly once.
Finding the equation of the circle given the centre and a point on the circumference.	You can find the distance between the two points to get the radius. And since you have your centre and radius, you can form an equation immediately.

5. Binomial Expansion

- If you're expanding $(x-a)^{\boldsymbol{n}}$ for some positive a, then you can check you've got your positives/negatives right, because your terms in the expansion should oscillate between positive and negative.
- Use bracketing to avoid BIDMAS type problems. e.g. For $(2-3x)^5$ you could write as $2^5+5(2)^4(-3x)+10(2)^3(-3x)^2+\cdots$
- You should remember that ${}^nC_0 = 1$, ${}^nC_1 = n$ and ${}^nC_2 = \frac{n(n-1)}{2}$
- Memorise the first five rows of Pascal's triangle. It'll save you time: If for example the power if 4, you should know the binomial coefficients will be 1, 4, 6, 4, 1 without having to determine them on a calculator.

6. Radians

Not much to say here! Just ensure you can INSTANTLY switch between radians and degrees for 30°, 45°, 60°, 90°, 180°, 270°, 360° . Obviously learn your formulae for arc length, sector area and segment area.

Thorin Radians Tip 1: Make sure your calculator is in RADIANS MODE whenever you're using sin/cos/tan.

Thorin Radians Tip 2: A number of people have asked me whether their calculator needs to be in radians mode when say using the expression $r\theta$ to find the arc length. The mode only affects use of trigonometric functions, not multiplication! Although of course, $r\theta$ only finds the arc length if your θ is in radians.

7. Geometric Sequences and Series

- Whenever you're asked to find the common ratio, and you have two available consecutive terms, just do the latter term divided by the one before it.
- The common ratio might be fraction or negative. If negative, the terms in your sequence will oscillate between positive and negative.
- Exam questions occasionally use variables for the values 'a' and 'r' instead of concrete numbers. Don't let this put you off: it's absolutely fine to have a common ratio of '$r = p$' or an initial value of '$a = 2z - 1$'.
- It might be helpful to write out your values of a, r and n before you put it into any formulae.
- When presenting your working, it's useful to use "$u_n =$" and "$S_n =$" to make clear to both yourself and examiner that you're finding the nth term or summing the first n terms. Don't get them (or their associated formulae) mixed up!
- I've seen exam papers ask for the proof that the sum of the first n terms of a geometric sequence is $\frac{a(1-r^n)}{1-r}$. The proof is in your textbook: just remember that you write out what S_n looks like, then rS_n, then subtract the latter from the former (which cancels all but the first and last terms in the addition).
- $\sum_{i=1}^{4} a_i$ for example just means you're summing the given expression where the value of i ranges between 1 and 4. i.e. $a_1 + a_2 + a_3 + a_4$.

8. Trigonometry

Although you have a calculator, it's helpful to learn your sin/cos/tan values for 30, 45, 60, 90, 180 of-by-heart. To remember them I 'picture' them in the following table:

	$0°$	$45°$	$90°$	$30°$	$60°$
sin	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
cos	1	$\frac{1}{\sqrt{2}}$	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
tan	0	1		$\frac{1}{\sqrt{3}}$	$\sqrt{3}$

- In the left block the only surd involved is $\sqrt{2}$ and in the right block just $\sqrt{3}$.
- For sin/cos for 30/60, they're all over 2. The diagonals from the top-left are the rational ones, and the other values the irrational ones.
- For tan in the right block, I remember that one is $\sqrt{3}$ and the other its reciprocal. But thinking about the graph of tan, $\tan(60) > \tan(30)$, so $\tan(30)$ is the smaller value of $\frac{1}{\sqrt{3}}$.
- $\sin(45)$ and $\cos(45)$ have the same value.
- To remember that $\sin(0) = 0$ and $\cos(0) = 1$, just picture the graphs of each. To work out $\cos(270)$ for example I 'trace' the graph out in my head, thinking about how it bobs up and down starting from the y-axis.

This 'memory technique' sounds absolutely nuts, but I promise you it works!

Solving Trigonometric Equations:

- Remember these '5 golden rules of angles' like the back of your hand (and hence you can completely avoid horrid 'cast' diagrams):
    1. $\sin(x) = \sin(180 - x)$
    2. $\cos(x) = \cos(360 - x)$
    3. sin and cos repeat every 360°
    4. tan repeats every 180°
    5. $\sin(x) = \cos(90 - x)$ (this is more of a C3 one)
- If your equation involves say sin and cos^2, change the squared term, i.e. the cos^2 to be consistent with the non-squared one. e.g. change $cos^2(2x)$ to $1 - sin^2(2x)$. This ensures you have a quadratic equation in terms of sin.
- The first thing you should do in any trig solvey question is **adjust the range**. e.g. If you're solving $\sin(2x) = \frac{1}{2}$ and $0 \leq x < 360$, then our adjusted range is $0 \leq 2x < 720$. Similarly if solving $3x - 20$, then $-20 \leq 3x - 20 < 1060$
- If you have a mixture of **<u>tan</u>** with sin or cos, express tan in terms of sin and cos, e.g. $tan(2x) = \frac{sin(2x)}{cos(2x)}$. Then you'd probably multiply the whole equation by $\cos(2x)$ and simplify before solving.
  Example: $\cos(x) - \tan(x) = 0 \;\rightarrow\; \cos^2(x) - \sin(x) = 0 \;\rightarrow\; 1 - \sin^2(x) - \sin(x) = 0$
  $\rightarrow \sin^2 x + \sin x - 1 = 0$
- It's useful to be able to add or subtract fractions in terms of π. e.g. $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$. Practice this if you're not comfortable with such manipulation.
- **Example**: Find all solutions to $\mathbf{\sin(2x - 45) = -\frac{1}{2}}$ **in the range** $\mathbf{0 \leq x \leq 360°}$.

$$-45 \leq 2x - 45 < 675$$
$$2x - 45 = \sin^{-1}\left(-\frac{1}{2}\right) = -30$$

  Using the two rules "$\sin(x) = \sin(180 - x)$" and "sin repeats every 360":

$$2x - 45 = -30, 210, 330, 570$$
$$So\ 2x = 15°, 255°, 375°, 615°$$
$$So\ x = 7.5°, 127.5°, 187.5°, 307.5°$$

- One last point: I've seen students work out alternative solutions as their last step, rather than when they do inverse sin/cos/tan. Suppose we want solutions in the range $0 \leq x < 180°$. **<u>The following is WRONG:</u>**

$$\sin(2x + 10) = \frac{\sqrt{3}}{2}$$
$$2x + 10 = 60$$
$$So\ x = 25°$$
$$And\ we\ also\ get\ 180° - 25° = 155°$$

Gollum's understandable reaction to this mathematical atrocity.

  The following is **<u>CORRECT</u>**:

$$10 \leq 2x + 10 < 370$$
$$\sin(2x + 10) = \frac{\sqrt{3}}{2}$$
$$2x + 10 = 60°, 120°, 420°$$
$$2x = 50°, 110°, 410°$$
$$x = 25°, 55°, \cancel{205°}$$

9. Differentiation

When asked to find the range of x for which a function is increasing/decreasing, you often get a quadratic inequality. Solve this in the same way as you did in C1: i.e. factorise and then SKETCH. Don't do anything stupid like going from $x^2 > 4$ to $x > \pm 2$ because you're just asking for a punch in the face. Instead, you should do $x^2 - 4 > 0$ (i.e. make one side of the inequality 0) then factorise to get $(x+2)(x-2) > 0$. Then you'd sketch to find that $x < -2 \text{ or } x > 2$.

Example: Find the values of x for which $f(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 - 2x$ is increasing.
It's increasing when $f'(x) > 0$. So differentiating: $2x^2 - 3x - 2 > 0$.
Factorising: $(2x+1)(x-2) > 0$. By sketching, we find that $x < -\frac{1}{2} \text{ or } x > 2$

Optimisation Problems: Always following the same structure:

1. Form two equations. For 3D shapes, this is usually one for the volume of a solid and the other for the surface area, and for 2D shapes, one for the area and one for the perimeter. You may need to introduce a variable yourself to represent some unknown.
   Your two equations will be in terms of two variables, one the 'constraint equation', and the other the equation you're trying to optimise.
2. You want your 'optimisation equation' in terms of one variable only. To do this, you substitute the other (constraint) equation into it.
3. You differentiate and set to 0 to find the minimum/maximum.
4. Finally, you put this value back into the equation you found in Step 2. Suppose for example you got $A = x^2 - 6x + 14$ as the equation for the surface area in terms of some length x. Then your optimum $x = 3$. Putting this back into the equation, we find that the optimum surface area is $A = 5$.
5. Often as a follow up question, you'll be asked to find whethe this is a minimum or maximum. Just differentiate again (to get the second order derivative). Remember that >0 means minimum, and <0 means maximum (I remember it as 'the opposite of what you might expect').
   $$\frac{dA}{dx} = 2x - 6 \qquad \frac{d^2A}{dx^2} = 2$$
   $2 > 0$ so we have a minimum.

Exam questions tend to be quite generous with these kinds of questions in terms of leading you to the answer: Steps 1/2 tend to be a 'Show that...' question. Remember that the volume of a cylinder is $\pi r^2 h$ and its surface area is $2\pi r^2 + 2\pi r h$ (i.e. the two ends and the shaft). You may also need your radian formulae for arc length and sector area.

Condition of student after C1 Integration.

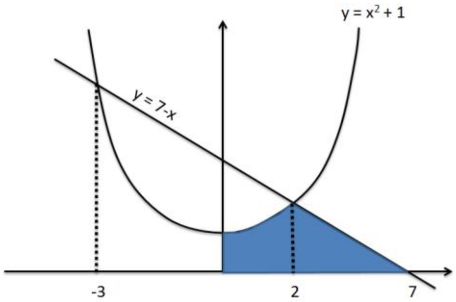

Condition of student after C2 Integration.


Condition of student after C3 Integration.

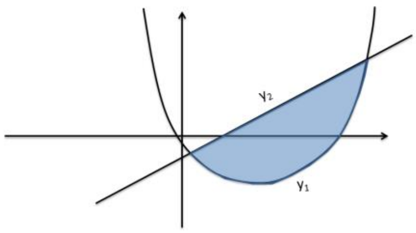

Condition of student after C4 Integration.

10. Integration

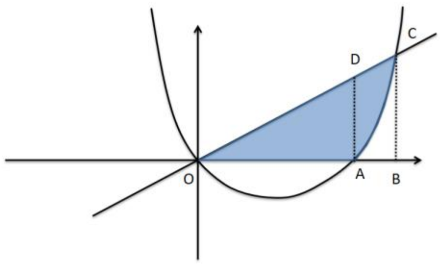
- I often see students integrating for their next line of working, but giving no notational indication that they've integrated. Be sure that for definite integration, you make use of square brackets.
- When substituting your limits into the integrated expression, make use of (normal) brackets to avoid minuses becoming pluses and vice-versa. Also be careful when one of the limits is negative. e.g.
$$[x^3 - 2x^2]_{-1}^{4} = (4^3 - 2 \cdot 4^2) - ((-1)^3 - 2(-1)^2)$$
Notice that I've used brackets both to: (a) keep the two expressions for my two different substitutions ($x = 4$ and $x = -1$) separate and (b) avoiding the usual safety hazards associated with cubing/squaring negative numbers.
If you have a silver calculator make use of the definite integration button!
- Areas below the x-axis will be negative. This is why you must separately find the area of different regions when the function goes both above and below the x-axis.
- Sometimes you need to use imaginative strategies to find areas by adding/subtracting regions. Suppose you were trying to find the shaded region in the following example:



One way you could do it is to find the area of the triangle between $x = 2$ and $x = 7$ (i.e. $\frac{1}{2} \times 5 \times 5$) and add the area under the curve between 0 and 2, i.e. $\int_0^2 x^2 + 1$. Alternatively, you could work out the area of the bigger triangle between x=0 and x=7 (i.e. $\frac{1}{2} \times 7 \times 7$), and cut out (i.e. subtract) the area between the line and curve, i.e. $\int_0^2 (7 - x) - (x^2 + 1) = \int_0^2 -x^2 - x + 6$. Or if you already knew the area of between the line and the curve between x=-3 and 2, you could always do the area of the big triangle between $x = -3$ and $x = 7$, cut out the area between the line and the curve, and also cut out the area under the curve between -3 and 0.
- Note that whenever you do the area between two lines, your area is guaranteed to be positive. And it doesn't matter if the area is both above and below the x-axis. E.g.



You'd just do $\int_a^b y_2 - y_1$. Remember you do the function for the TOP line minus the function for the BOTTOM line in this region.
- Be careful when you haven't got the full region between a line and a curve. For example:



In such a case we could either: (a) Find the area of triangle OAD and add the area between the line and curve between x=A and x=B or (b) find the area of the triangle OBC, and subtract the area under the curve between x=A and x=B. In general, just be conscientious what areas you're adding and subtracting.