BMOS MENTORING SCHEME (Intermediate Level)
October 2010 (Sheet 1)
Comments for Mentors and Teachers

Welcome to the 2010/11 Intermediate Mentoring Scheme and thank you very much for your part in this.

Just to let you know how the Intermediate Scheme works. Each month I set a sheet of 8 questions graded from "quite approachable" (Q1,2 approx), to "medium difficulty" (Q3-6) to "hard" (Q7,8). All this is approximate and will vary from month to month according to the questions I manage to get together. I will also send out to mentors/teachers a sheet of Comments and Hints, with the questions or shortly afterwards. A deadline will be set near to the end of the month for mentees to submit work. After this, a sheet with more formal solutions or outline solutions with comments on how to approach such problems will be sent to all the mentees. If you wish to give feedback on the sheets - whether questions are the right level etc, do please do so to me on ra@oundleschool.org.uk or to mentoring@ukmt.org.uk . Please forgive me if I fail to respond to all of these (or if I have failed in the past). I get too busy sometimes and things fall by the wayside! But please don't be put off emailing as I do welcome helpful comments and constructive criticism!

The role of the mentor is crucial, discussing approaches, giving hints etc and encouraging mentees. But we must not give away the secrets too easily! Otherwise the mentees will not gain the maximum from doing the questions, but of course we must help. And in order to help, ideally we should have a go at the questions ourselves, although I am conscious that often we are very busy. So the point of my Hints and Comments sheet is to help mentors give some clues without giving the whole game away and thereby taking away their fun. I also don't want to spoil your fun, but I do want to help. The full solutions will follow in 10 days or so (for mentors only!) and then they can go to mentees after they have submitted their solutions.

I have tried to make this first sheet accessible to encourage them to have a go at all the questions, but also to build in enough difficulty to make it challenging.

Richard Atkins October 1st 2010.

Q1. I suppose mentees have to be encouraged to do the most obvious thing, even if they don't know where that will lead. In this case the most obvious thing is to square up. And then do it again! And again!! So we get to the fact that $a^7 = b^8$. Now they need to think what this means: that since we are working in integers, a must be the 8th power of an integer greater than 1, and b must be a 7th power. Since we are looking for the least possible value of $a + b$, we choose $a = 2^8$ and $b = 2^7$. So $a + b = 256 + 128 = 384$.

Q2. One key idea for people starting to do problem-solving is to use algebra, and this may not be instinctive to someone new to the mentoring schemes. So here is first almost ubiquitous idea for solving problems. Introduce some letters to stand for things!! Not too many, of course. Here they should ideally use two letters. If they are still puzzled, suggest that they suppose that Ellie has done n tests before the last two, and that her average score up to that point was m. [Of course, they could use slightly different quantities, eg the total number of tests.]
They should then translate the two pieces of information into two equations. The first becomes:
$\frac{mn+98}{n+1} = m+1$ and the second $\frac{mn+98+70}{n+2} = m+1-2$. These two equations simplify to $m + n = 97$ and $2m - n = 170$ which are easily solved to give $m = 89$ and $n = 8$, so therefore she sat 10 tests in total.

Q3. This pair of equations may be the first time some people have come across simultaneous equations which are rather more complicated than the standard ones. So they need to be encouraged to look at them and think what they could do perhaps to get a quadratic in something other than just x or y. They should think how various algebraic quantities are related and could be given the hint to remember that $(x + y)^2 = x^2 + 2xy + y^2$. So one idea (the most obvious to me) is to move the xy to the other side in the second equation and square up. If they then multiply out each side and substitute 25 for $x^2 + y^2$ in the LHS. Then they will get a quadratic in xy which can be solved. Or alternatively double the second equation (to involve $2xy$) and then add to the first to give $x^2 + y^2 + 2x + 2y + 2xy = 25 + 2 \times 19$. This then gives a quadratic in $(x + y)$ this time, namely: $(x + y)^2 + 2(x + y) - 63 = 0$. Factorising gives $(x + y - 7)(x + y + 9) = 0$. Now we consider the two possibilities:

If $x + y = 7$, equation (2) gives $xy = 12$, and substituting in (1) gives $x^2 + \dfrac{144}{x^2} = 25$ so $x^4 - 25x^2 + 144 = 0$.

Hence $(x^2 - 9)(x^2 - 16) = 0$, so $x^2 = 9$ or 16 and hence $x = \pm 3$ or ± 4. Since $xy > 0$, both x and y must have the same sign, but since they add to 7, they must both be positive. So the solution pairs are (3, 4) and (4, 3).

Now for the second case:

If $x + y = -9$, equation (2) gives $xy = 28$ and substituting in (1) gives $x^2 + \dfrac{784}{x^2} = 25$ so $x^4 - 25x^2 + 784 = 0$.

Since $b^2 - 4ac < 0$, this equation yields no other solutions.

Q4. Well I suppose the first thing to do is to start messing around and trying it out. Perhaps draw up a 5×5 square and cut out a 3×1 domino and see how they could be placed. But it is really difficult to know what to do from here. Hopefully some really inspired people who can think creatively might come up with the idea which is really the way to approach such problems, and that is to use some sort of colouring argument. The idea here is to colour the grid chequer-board style, but in this case using 3 colours.

If I knew how to do this quickly on a grid I would, but since I don't I will do it with numbers!

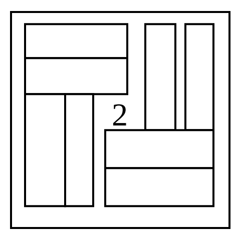
Suppose the colours are 1, 2, 3. If we colour the grid as shown on the right, we see that colours 1 and 3 both occur 8 times, whereas 2 occurs 9 times. Since a 3×1 domino must cover each colour once, the coin therefore can only be on one of the squares coloured (2). Clearly not all of these are possible, as one can clearly see by trying one of the 2s on the top left.

1	2	3	1	2
2	3	1	2	3
3	1	2	3	1
1	2	3	1	2
2	3	1	2	3

Now colour using the other diagonals:

1	2	3	1	2
3	1	2	3	1
2	3	1	2	3
1	2	3	1	2
3	1	2	3	1

This time colour (1) is the one that occurs 9 times, so the coin can only be on one of the squares coloured (1). The only one of these which is one of the (2) squares before is the centre square, so this is the only possible one. However...! We are not yet done! We must show that this is possible. The diagram on the right confirms this.



Q5. Of course the first thing to do is to draw a diagram (using compass, pencil and ruler!). If one then draws perpendiculars from the two circles furthest from A, there are two similar triangles, so the distance of the perpendicular from the centre of the middle circle is 9. It just then needs an application of Pythagoras to deduce that $EF = 24$.

Q6. A positive integer has a remainder of 1 when divided by any of the integers from 2 to 11 if and only if the integer is of the form $mt + 1$, where t is a non-negative integer and m is the least common multiple of the integers from 2 to 11. This number is $2^3 \times 3^2 \times 5 \times 7 \times 11$ which is 27720. Therefore consecutive integers with the desired property differ by 27720.

Q7. Let k denote the number of matches in which women defeated men, and W and M denote the total number of matches won by women and men respectively. The total number of matches, the number of matches between a man and a woman, the number of matches between two men and the number of matches between two women are

$$\frac{3n(3n-1)}{2}, \quad 2n^2, \quad \frac{2n(2n-1)}{2}, \text{ and } \frac{n(n-1)}{2} \text{ respectively.}$$

So $$\frac{M+W}{W} = \frac{3n(3n-1) \div 2}{k + n(n-1) \div 2} \quad \text{and also} \quad \frac{M+W}{W} = \frac{M}{W} + 1 = \frac{12}{7} \quad \text{from what we are told.}$$

Equating these things leads to the equation $63n^2 - 21n = 24k + 12n^2 - 12n$

so $k = \dfrac{17n^2 - 3n}{8}$. But since $k \le 2n^2$, we have $17n^2 - 3n \le 16n^2$, so $n \le 3$.

Substituting $n = 1, 2, 3$ gives $k = 14/8$, $62/8$, $144/8$. Since k must be an integer, $n = 3$.

Q8. The key idea here is that k divides $km + n$ if and only if k divides n.
[The phrase "k divides n" means "k is a factor of n".]

The effect of this is that one can try to rewrite the condition in some way as to make $7N + 55$ divide some integer.

So $7N + 55$ divides $N^2 - 71$ if and only if it divides $7N^2 - 497$ since 7 is coprime to $7N + 55$ and $7N^2 - 497 = N(7N + 55) - 55N - 497$. So $7N + 55$ needs to divide $55N + 497$ which is true if and only if it divides $7 \times 55N + 7 \times 497$. But this is $55(7N + 55) - 55 \times 55 + 7 \times 497$.
So $7N + 55$ needs to divide $7 \times 497 - 55 \times 55$, which is $3479 - 3025 = 454$.
Now the factors of 454 are only $\pm1, \pm2, \pm227, \pm454$.
Since $7N + 55$ is one less than a multiple of 7 (or more succinctly $7N + 55 \equiv -1 \bmod 7$) we are only interested in which of these eight numbers is one less than a multiple of 7.
The only one is 454, so $7N + 55 = 454$ and hence there is only one possible value of N, namely 57.

I hope these comments are helpful and that your mentees enjoy doing the sheet. If you do have any comments either on the problems or the hints or the solutions which help me to target subsequent ones, a brief email would be great. Feedback to mentoring@ukmt.org is of course also very welcome.