BMOS MENTORING SCHEME (Intermediate Level)
October 2010 (Sheet 1)
Solutions and Explanatory Notes

Welcome to the 2010/11 Intermediate Mentoring Scheme. In general the questions on each sheet are graded approximately as follows: "quite approachable" (Q1,2 approx), to "medium difficulty" (Q3-6) to "hard" (Q7,8).

I hope you will find that the problems are enjoyable to tackle, challenging and rewarding. The methods will vary; a good problem is one where you say "I have no idea how to do that. Let's try a few things and see what the problem is all about." You try some small cases, some special cases, see if there are any obvious values which work, etc and then hopefully you come up with a method of solution. That is what you write up, though you can include some of your early work if you like, but it is not an integral part of the solution. Please contact you mentor/teacher a few times during the month to ask for hints or just say how you are doing.

Richard Atkins October 17th 2010.

Q1. Generally as you approach a problem I would encourage you to do the most obvious thing, even if you don't know where that will lead. In this case the most obvious thing is to square up. And then do it again! And again!! So we get to the fact that $a^7 = b^8$. Now you need to think what this means: that since we are working in integers, a must be the 8th power of an integer greater than 1, and b must be a 7th power. Since we are looking for the least possible value of $a + b$, we choose $a = 2^8$ and $b = 2^7$. So $a + b = 256 + 128 = 384$.

Q2. One key idea as you start to do problem-solving is to use algebra, and this may not be instinctive to you if you are new to the mentoring schemes. So here is first almost ubiquitous idea for solving problems. Introduce some letters to stand for things!! Not too many letters - generally as few as possible. Here I suggest you use two letters. Make sure you say what they stand for, partly to get it clear in your own mind, partly because without this it makes no sense to the reader!

Possible solution:

Suppose that Ellie has done n tests before the last two, and that her average score up to that point was m.
So her total rose from mn to $mn + 98$ after $n + 1$ tests, raising her average to $m + 1$.
So $mn + 98 = (m + 1)(n + 1)$. After the last test, her total rose to $mn + 98 + 70$ and this gave an average of $m - 1$ after $n + 2$ tests. So we have a second equation $mn + 98 + 70 = (n + 2)(m - 1)$.

$$mn + 98 = (m + 1)(n + 1) \quad \Rightarrow \quad mn + 98 = mn + m + n + 1 \quad \Rightarrow \quad m + n = 97.$$

And $$mn + 98 + 70 = (n + 2)(m - 1) \quad \Rightarrow \quad mn + 168 = mn + 2m - n - 2 \quad \Rightarrow \quad 2m - n = 170.$$

Solving these gives $m = 89$ and $n = 8$, so therefore she sat 10 tests in total.

Q3. This pair of equations may be the first time you have come across simultaneous equations which are rather more complicated than the standard ones. You need to look at them and think what you could do perhaps to get a quadratic equation in something other than just x or y. You need to think how various algebraic quantities are related and remember that $(x + y)^2 = x^2 + 2xy + y^2$. So one idea (the most obvious to me) is to move the xy to the other side in the second equation and square up.

Possible solution:

$$x^2 + y^2 = 25 \qquad (1)$$
$$x + y + xy = 19 \qquad (2)$$

From (2), we get $$x + y = 19 - xy$$
Squaring: $$(x + y)^2 = (19 - xy)^2,$$
$$\Rightarrow \quad x^2 + 2xy + y^2 = 361 - 38xy + x^2y^2$$
Since $x^2 + y^2 = 25$, $$2xy + 25 = 361 - 38xy + x^2y^2,$$
so $$0 = x^2y^2 - 40xy + 336.$$
Factorising gives $$0 = (xy - 12)(xy - 28), \qquad \text{so} \qquad xy = 12 \text{ or } 28.$$

Case 1: $\quad$ ***xy* = 12** $\quad$ If this is true, then $y = \dfrac{12}{x}$ so substituting in (1) gives $x^2 + \dfrac{144}{x^2} = 25$

so $x^4 - 25x^2 + 144 = 0$.

Factorising gives $(x^2 - 9)(x^2 - 16) = 0$,

so $x^2 = 9$ or 16

and hence $x = \pm 3$ or ± 4

Since $xy = 12$, possible pairings for (x, y) would be (3, 4), (–3, –4), (4, 3) and (–4, –3)
But since $x + y = 19 - xy$, $x + y = 7$, so the only possible solutions in this case are (3, 4) and (4, 3).

Case 2: $\quad$ ***xy* = 28** $\quad$ If this is true, then $y = \dfrac{28}{x}$ so substituting in (1) gives $x^2 + \dfrac{784}{x^2} = 25$

so $x^4 - 25x^2 + 784 = 0$.

This would imply that $(x^2 - 12.5)^2 = 12.5^2 - 784$, which is negative, so this is impossible.

Therefore the only possible solutions are (3, 4) and (4, 3).

[*Note*: This question needs quadratic equations which may not have been done in school for younger mentees. Hopefully the working above will show you how such equations can be solved by factoring. In general the equation $ax^2 + bx + c = 0$ has either 2, 1 or 0 solutions according to whether the quantity $b^2 - 4ac$ is either >0, $=0$ or <0. The quantity $b^2 - 4ac$ is called the discriminant and this can be used quickly to show that there are no solutions in Case 2 above.]

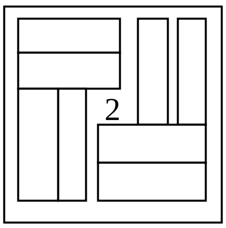
Q4. Well I suppose the first thing to do is to start messing around and trying it out. Perhaps draw up a 5 × 5 square and cut out a 3 × 1 domino and see how they could be placed. But it is really difficult to know what to do from here. If you are really inspired, you might come up with the idea which is really the way to approach such problems, and that is to use some sort of colouring argument. The idea here is to colour the grid chequer-board style, but in this case using 3 colours.
Instead of colouring, I will do this with numbers!
Suppose the colours are 1, 2, 3. If we colour the grid as shown on the right, we see that colours 1 and 3 both occur 8 times, whereas 2 occurs 9 times.
Since a 3 × 1 domino must cover each colour once, the coin therefore can only be on one of the squares coloured (2). Clearly not all of these are possible, as one can clearly see by trying one of the 2s on the top left.

1	2	3	1	2
2	3	1	2	3
3	1	2	3	1
1	2	3	1	2
2	3	1	2	3

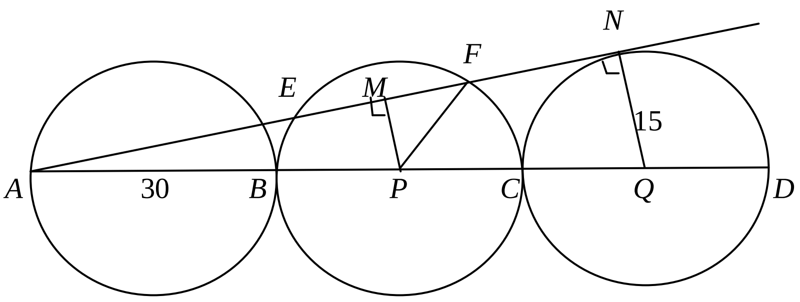
Now colour using the other diagonals:

1	2	3	1	2
3	1	2	3	1
2	3	1	2	3
1	2	3	1	2
3	1	2	3	1

This time colour (1) is the one that occurs 9 times, so the coin can only be on one of the squares coloured (1).
The only one of these which is one of the (2) squares before is the centre square, so this is the only possible one. However...! We are not yet done!
We must show that this is possible.
The diagram on the right confirms this.



Q5. Firstly, please do us all a big favour and draw a **decent** diagram (using compass, pencil and ruler!).



Very often the key to a geometry problem is to add something to the diagram to enable you to make some deductions. The question is what to add. Try to avoid adding lots of things as this can make everything very confusing. I suggest the key thing here is to construct two perpendiculars from the centres *P* and *Q* as shown. We know the radii are all 15, so *NQ* = 15. Now notice that Δs *AMP* and *ANQ* are similar.
So we write $\Delta AMP \sim \Delta ANQ$. So the ratio *MP* : *NQ* = *AP* : *AQ* = 45 : 75 = 3 : 5. So *MP* = 9.
Now draw *PF*. By Pythagoras $FM = \sqrt{15^2 - 9^2} = 12$. [Or just say, ΔPMF is a 3,4,5 Δ, so *FM* = 12.]
Therefore the length *EF* = 24 since the perpendicular from the centre bisects the chord *EF*.

Q6. A positive integer has a remainder of 1 when divided by any of the integers from 2 to 11 if and only if the integer is of the form *mt* + 1, where *t* is a non-negative integer and *m* is the least common multiple of the integers from 2 to 11. This number is $2^3 \times 3^2 \times 5 \times 7 \times 11$ which is 27720. Therefore consecutive integers with the desired property differ by 27720.

Q7. Let *k* denote the number of matches in which women defeated men, and *W* and *M* denote the total number of matches won by women and men respectively. The total number of matches, the number of matches between a man and a woman, the number of matches between two men and the number of matches between two women are

$$\frac{3n(3n-1)}{2}, \quad 2n^2, \quad \frac{2n(2n-1)}{2}, \text{ and } \frac{n(n-1)}{2} \text{ respectively.}$$

So $\dfrac{M+W}{W} = \dfrac{3n(3n-1) \div 2}{k + n(n-1) \div 2}$ and also $\dfrac{M+W}{W} = \dfrac{M}{W} + 1 = \dfrac{12}{7}$ from what we are told.

Equating these things leads to the equation $63n^2 - 21n = 24k + 12n^2 - 12n$

so $k = \dfrac{17n^2 - 3n}{8}$. But since $k \le 2n^2$, we have $17n^2 - 3n \le 16n^2$, so $n \le 3$.
Substituting *n* = 1, 2, 3 gives *k* = 14/8, 62/8, 144/8. Since *k* must be an integer, *n* = 3.

Q8. The key idea here is that k divides $km + n$ if and only if k divides n.
[The phrase "k divides n" means "k is a factor of n".]

The effect of this is that one can try to rewrite the condition in some way as to make $7N + 55$ divide some integer.

So $7N + 55$ divides $N^2 - 71$ if and only if it divides $7N^2 - 497$ since 7 is coprime to $7N + 55$ and $7N^2 - 497 = N(7N + 55) - 55N - 497$. So $7N + 55$ needs to divide $55N + 497$ which is true if and only if it divides $7 \times 55N + 7 \times 497$. But this is $55(7N + 55) - 55 \times 55 + 7 \times 497$.
So $7N + 55$ needs to divide $7 \times 497 - 55 \times 55$, which is $3479 - 3025 = 454$.
Now the factors of 454 are only ±1, ±2, ±227, ±454.
Since $7N + 55$ is one less than a multiple of 7 (or more succinctly $7N + 55 \equiv -1 \bmod 7$) we are only interested in which of these eight numbers is one less than a multiple of 7.
The only one is 454, so $7N + 55 = 454$ and hence there is only one possible value of N, namely 57.

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