UK JUNIOR MATHEMATICAL OLYMPIAD

2007 to 2010

Organised by the

United Kingdom Mathematics Trust

UK Junior Mathematical Olympiad
2007 to 2010

Question Papers and Solutions

Organised by the **United Kingdom Mathematics Trust**

Contents

Background

The Junior Mathematical Olympiad (JMO) is the follow-up competition for pupils (aged up to about 13) who do extremely well in the UKMT Junior Mathematical Challenge (about 1 in 200 are invited to take part). The JMO was established in 1989 and, for a more in-depth treatment of its early years and relevant mathematics, the book *More Mathematical Challenges* by Tony Gardiner (Cambridge University Press, 1997, ISBN 0-521-58568-6) is strongly recommended. A JMO paper has two sections. Section A contains a number of relatively short questions which have easily identifiable answers. Section B contains fewer, longer questions to which full written solutions are required. (Note that very few pupils will satisfactorily complete more than one or two questions from Section B in the examination.) It is hoped that those using this booklet make a sustained attempt at the questions before turning to the solutions.

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RULES AND GUIDELINES :
READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators *and measuring instruments* is forbidden.**
3. All candidates must be in *School Year 8 or below* (England and Wales), *S2 or below* (Scotland), *School Year 9 or below* (Northern Ireland).
4. For questions in Section A *only the answer is required*. Enter each answer neatly in the relevant box on the Front Sheet. Do not hand in rough work.

   For questions in Section B you must give *full written solutions*, including clear mathematical explanations as to why your method is correct.

   Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Front Sheet on top.

   Do not hand in rough work.
5. Questions A1-A10 are relatively short questions. Try to complete Section A within the first hour so as to allow at least one hour for Section B.
6. Questions B1-B6 are longer questions requiring *full written solutions*.
   This means that each answer must be accompanied by clear explanations and proofs.
   Work in rough first, then set out your final solution with clear explanations of each step.
7. These problems are meant to be challenging! Do not hurry. Try the earlier questions in each section first (they tend to be easier). Try to finish whole questions even if you can't do many. A good candidate will have done most of Section A and given solutions to at least two questions in Section B.
8. Answers must be FULLY SIMPLIFIED, and EXACT using symbols like π, fractions, or square roots if appropriate, but NOT decimal approximations.

DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

2007

Section A

A1 What is the value of $1^5 - 2^4 + 3^3 - 4^2 + 5^1$?

A2 What is the value of k if "$7k$ minutes past nine" is the same time as "$8k$ minutes to ten"?

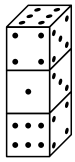
A3 Charlie boils seven eggs for his breakfast. He puts the eggs into the pan one at a time, but waits one minute after putting one egg in before putting the next egg in. If he boils each egg for three minutes, how long does the whole operation take from the moment he puts the first egg in to the moment he takes the seventh egg out?

A4 The hobbits Frodo, Sam, Pippin and Merry have breakfast at different times. Each one takes a quarter of the porridge in the pan, thinking that the other three have not yet eaten. What fraction of the porridge is left after all four hobbits have had their breakfast?

A5 The diagram shows a tower consisting of three identical dice.

On these dice, each pair of opposite faces has a total of seven dots.

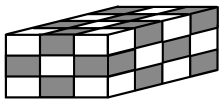
How many dots are there on the face on which the tower stands?



A6 The sizes in degrees of the interior angles of a pentagon are consecutive whole numbers. What is the size of the largest of these angles?

A7 A large cuboid is made from cuboids of equal size, coloured alternately black and white, as shown.

What fraction of the surface area of the large cuboid is black?



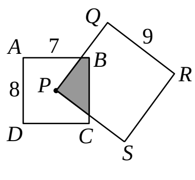
A8 Pegs numbered 1 to 50 are placed in order in a line with number 1 on the left. They are then knocked over one at a time following these rules:

- Of the pegs which are still standing, knock down alternate ones, starting with the first peg on the left.
- Each time you reach the end of the row, repeat the previous rule.

What is the number of the last peg to be knocked over?

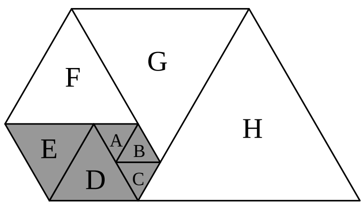
A9 The diagram shows squares $ABCD$ and $PQRS$ of side length 8 units and 9 units respectively. Point P is the centre of square $ABCD$; PQ intersects AB at a point 7 units from A.

What is the perimeter of the shaded region?



A10 The diagram shows a spiral of equilateral triangles. After the first five triangles A, B, C, D, E (shown shaded), the next triangle is always placed alongside two others: the one placed immediately before and one placed earlier. The smallest triangles have sides of length 1 unit.

What is the length of the sides of the fifteenth triangle?



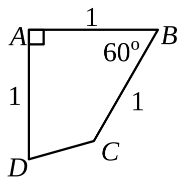
Section B

Your solutions to Section B will have a major effect on the JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief 'answers').

B1 Find four integers whose sum is 400 and such that the first integer is equal to twice the second integer, three times the third integer and four times the fourth integer.

B2 The diagram shows a quadrilateral $ABCD$ in which AB, BC and AD are all of length 1 unit, $\angle BAD$ is a right angle and $\angle ABC$ is $60°$.

Prove that $\angle BDC = 2 \times \angle DBC$.

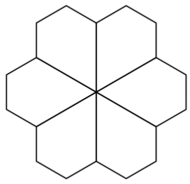


B3 (a) Yesterday evening, my journey home took 25% longer than usual.
By what percentage was my average speed reduced compared to normal?

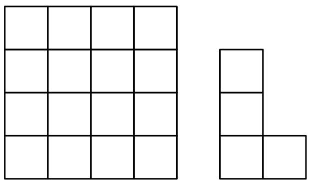
(b) By what percentage would I need to increase my usual average speed in order for the journey to take 20% less time than usual?

B4 Find a rule which predicts exactly when the sum of five consecutive integers is divisible by 15.

B5 A window is constructed of six identical panes of glass. Each pane is a pentagon with two adjacent sides of length two units.
The other three sides of each pentagon, which are on the perimeter of the window, form half of the boundary of a regular hexagon
Calculate the exact area of glass in the window.



B6 We want to colour red some of the cells in the 4×4 grid shown so that wherever the L-shaped piece is placed on the grid it covers at least one red cell. The L-shaped piece may only cover complete cells, may be rotated, but may not be turned over and may not extend beyond the grid.



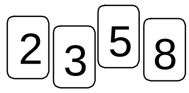
(a) Show that it is possible to achieve this by colouring exactly four cells red.

(b) Show that it is impossible to achieve this by colouring fewer than four cells red.

2008

Section A

A1 In how many ways is it possible to place side by side two of the cards shown to form a two-digit prime number?



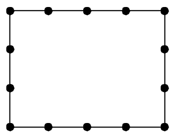
A2 Tony wants to form a square with perimeter 12 cm by folding a rectangle in half and then in half again. What is the maximum possible perimeter of the original rectangle?

A3 Given that $\frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{18} + \frac{1}{x} = 1$, what is the value of x ?

A4 How many three-digit numbers have the product of their digits equal to 6?

A5 A 3 by 4 rectangle has 14 points equally spaced around its four sides, as shown.

In how many ways is it possible to join two of the points by a straight line so that the rectangle is divided into two parts which have areas in the ratio 1 : 3?



A6 How many positive square numbers are factors of 1600?

A7 In a *Magic Square*, the sum of the three numbers in each row, each column and each of the two main diagonals is the same.

What is the value of x in the partially completed magic square shown?

		6
x	4	5

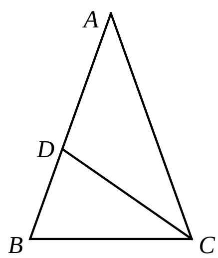
A8 Granny shares a packet of sweets between her four granddaughters. The girls, Clarrie, Lizzie, Annie and Danni, always in that order, each take 8 sweets in turn, over and over again until, finally, there are some sweets left for Danni, but there are fewer than 8. Danni takes all the sweets that are left. The other three girls then give Danni some of their sweets so that all four girls have the same number of sweets.

How many sweets does each of the other three granddaughters give to Danni?

A9 In the diagram, CD is the bisector of angle ACB.

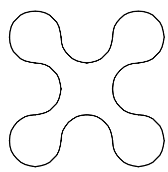
Also, $BC = CD$ and $AB = AC$.

What is the size of angle CDA?



A10 The perimeter of the shape shown on the right is made from 20 quarter-circles, each with radius 2 cm.

What is the area of the shape?



Section B

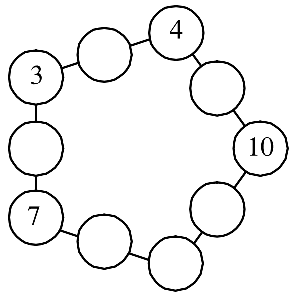
Your solutions to Section B will have a major effect on your JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief 'answers').

B1 Tamsin has a selection of cubical boxes whose internal dimensions are whole numbers of centimetres, that is, $1\text{ cm} \times 1\text{ cm} \times 1\text{ cm}$, $2\text{ cm} \times 2\text{ cm} \times 2\text{ cm}$, and so on.

What are the dimensions of the smallest of these boxes in which Tamsin could fit ten rectangular blocks each measuring $3\text{ cm} \times 2\text{ cm} \times 1\text{ cm}$ without the blocks extending outside the box?

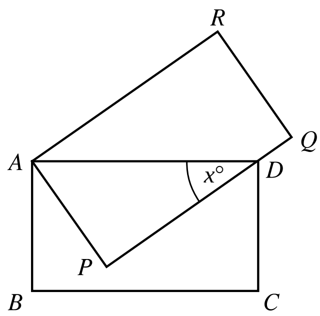
B2 Each of the numbers from 1 to 10 is to be placed in the circles so that the sum of each line of three numbers is equal to T. Four numbers have already been entered.

Find all the possible values of T.



B3 In the diagram *ABCD* and *APQR* are congruent rectangles. The side *PQ* passes through the point *D* and $\angle PDA = x°$.

Find an expression for $\angle DRQ$ in terms of x.

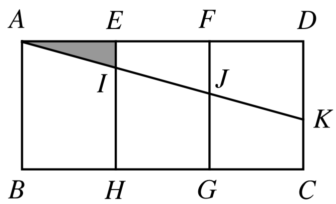


B4 For each positive two-digit number, Jack subtracts the units digit from the tens digit; for example, the number 34 gives $3 - 4 = -1$.

What is the sum of all his results?

B5 In the diagram, the rectangle *ABCD* is divided into three congruent rectangles. The line segment *JK* divides *CDFG* into two parts of equal area.

What is the area of triangle *AEI* as a fraction of the area of *ABCD*?



B6 In a sequence of positive integers, each term is larger than the previous term. Also, after the first two terms, each term is the sum of the previous two terms.

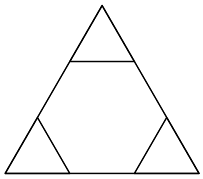
The eighth term of the sequence is 390. What is the ninth term?

2009

Section A

A1 What is the value of $200^2 + 9^2$?

A2 The diagram shows a regular hexagon inside an equilateral triangle. The area of the larger triangle is $60\ \text{cm}^2$. What is the area of the hexagon?



A3 The positive whole numbers a, b and c are all different and $a^2 + b^2 + c^2 = 121$. What is the value of $a + b + c$?

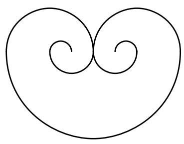
A4 The sum of three numbers is 2009. The sum of the first two numbers is 1004 and the sum of the last two is 1005. What is the product of all three numbers?

A5 Andrea's petrol tank holds up to 44 litres of fuel. She goes to the garage when her tank is a quarter full and puts more petrol in the tank until it is two-thirds full. How many litres of petrol does she put in the tank?

A6 The shorter sides of a right-angled isosceles triangle are each 10 cm long. The triangle is folded in half along its line of symmetry to form a smaller triangle. How much longer is the perimeter of the larger triangle than that of the smaller?

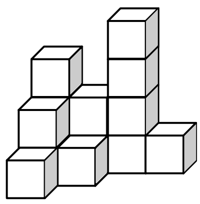
A7 Dean runs on a treadmill for thirty minutes. To keep his mind active as well as his legs, he works out what fraction of the total time has passed at each half minute and minute from the start. How many of the results of his calculations can be expressed in the form $\frac{1}{n}$, where n is an integer greater than 1?

A8 The diagram shows a curve made from seven semicircular arcs, the radius of each of which is 1 cm, 2 cm, 4 cm or 8 cm. What is the length of the curve?



A9 A book has 89 pages, but the page numbers are printed incorrectly. Every third page number has been omitted, so that the pages are numbered 1, 2, 4, 5, 7, 8, … and so on. What is the number on the last printed page?

A10 Gill piles up fourteen bricks into the shape shown in the diagram. Each brick is a cube of side 10 cm and, from the second layer upwards, sits exactly on top of the brick below.
Including the base, what is the surface area of Gill’s construction?



Section B

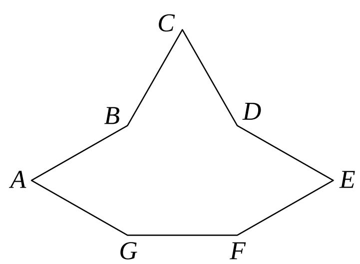
Your solutions to Section B will have a major effect on your JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief 'answers').

B1 In 2007 Alphonse grew twice the number of grapes that Pierre did. In 2008 Pierre grew twice the number of grapes that Alphonse did. Over the two years Alphonse grew 49 000 grapes, which was 7600 less than Pierre. How many grapes did Alphonse grow in 2007?

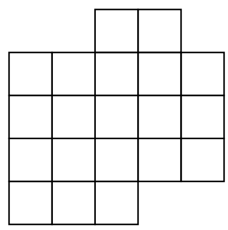
B2 $ABCD$ is a square. The point E is outside the square so that CDE is an equilateral triangle. Find angle BED.

B3 Tom left a motorway service station and travelled towards Glasgow at a steady speed of 60 mph. Tim left the same service station 10 minutes after Tom and travelled in the same direction at a steady speed, overtaking Tom after a further 1 hour 40 minutes. At what speed did Tim travel?

B4 The diagram shows a polygon $ABCDEFG$, in which $FG = 6$ and $GA = AB = BC = CD = DE = EF$. Also $BDFG$ is a square. The area of the whole polygon is exactly twice the area of $BDFG$.
Find the length of the perimeter of the polygon.



B5 An ant wishes to make a circuit of the board shown, visiting each square exactly once and returning to the starting square. At each step the ant moves to an adjacent square across an edge. Two circuits are considered to be the same if the first follows the same path as the second but either starts at a different square or follows the same path in reverse. How many such circuits are possible?



B6 I want to choose a list of n different numbers from the first 20 positive integers so that no two of my numbers differ by 5. What is the largest value of n for which this is possible? How many different lists are there with this many numbers?

2010

Section A

A1 What is the value of $\frac{1}{1} + \frac{2}{\frac{1}{2}} + \frac{3}{\frac{1}{3}} + \frac{4}{\frac{1}{4}} + \frac{5}{\frac{1}{5}}$?

A2 Given that $x : y = 1 : 2$ and $y : z = 3 : 4$, what is $x : z$?

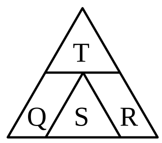
A3 Tom correctly works out 20^{10} and writes down his answer in full.
How many digits does he write down in his full answer?

A4 Three monkeys Barry, Harry and Larry met for tea in their favourite café, taking off their hats as they arrived. When they left, they each put on one of the hats at random. What is the probability that none of them left wearing the same hat as when they arrived?

A5 The sum of two positive integers is 97 and their difference is 37. What is their product?

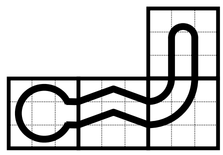
A6 In the diagram, the equilateral triangle is divided into two identical equilateral triangles S and T, and two parallelograms Q and R which are mirror images of each other.

What is the ratio of area R : area T ?



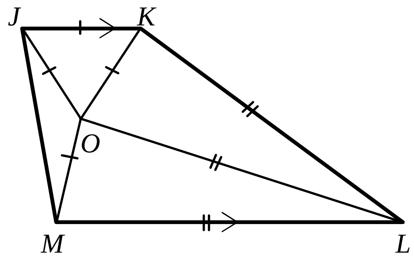
A7 What is the largest possible angle in an isosceles triangle, in which the difference between the largest and smallest angles is 6°?

A8 The four square tiles having the designs as shown can be arranged to create a closed loop.
How many distinct closed loops, including the one shown here, can be made from the tiles?
(The tiles may be rotated, but a rotation of a loop is not considered distinct. A loop need not use all four tiles and may not use more than one of each type).



A9 Abbie, Betty and Clara write names on bookmarks sold for charity.
Abbie writes 7 names in 6 minutes, Betty writes 18 names in 10 minutes and Clara writes 23 names in 15 minutes.
If all of the girls work together at these rates, how long will it take them to write 540 names?

A10 In the diagram, JK and ML are parallel, $JK = KO = OJ = OM$ and $LM = LO = LK$.
Find the size of angle JMO.



Section B

Your solutions to Section B will have a major effect on your JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief ‘answers’).

B1 In a sequence of six numbers, every term after the second term is the sum of the previous two terms. Also, the last term is four times the first term, and the sum of all six terms is 13.

What is the first term?

B2 The eight-digit number “*ppppqqqq*”, where p and q are digits, is a multiple of 45.

What are the possible values of p?

B3 Jack and Jill went up a hill. They started at the same time, but Jack arrived at the top one-and-a-half hours before Jill. On the way down, Jill calculated that, if she had walked 50% faster and Jack had walked 50% slower, then they would have arrived at the top of the hill at the same time.

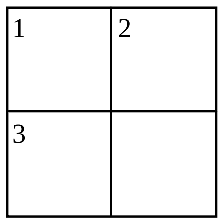
How long did Jill actually take to walk up to the top of the hill?

B4 The solution to each clue of this crossnumber is a two-digit number, not beginning with zero.

In how many different ways can the crossnumber be completed correctly?

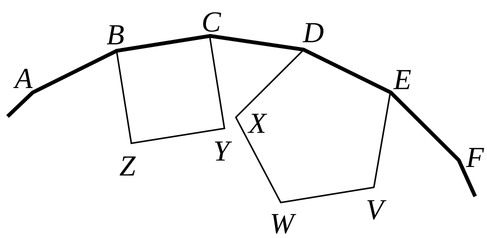
Clues
Across
1. A triangular number
3. A triangular number

Down
1. A square number
2. A multiple of 5



B5 The diagram shows part of a regular 20-sided polygon (an icosagon) *ABCDEF*…, a square *BCYZ* and a regular pentagon *DEVWX*.

Show that the vertex X lies on the line DY.



B6 Sam has put sweets in five jars in such a way that no jar is empty and no two jars contain the same number of sweets. Also, any three jars contain more sweets in total than the total of the remaining two jars.

What is the smallest possible number of sweets altogether in the five jars?

2007 Solutions

A1 **1** $1^5 - 2^4 + 3^3 - 4^2 + 5^1 = 1 - 16 + 27 - 16 + 5 = 33 - 32 = 1$.

A2 **4** If '$7k$ minutes past nine' is the same time as '$8k$ minutes to ten' then $7k + 8k = 60$, so $k = 4$. (*The two times are 28 minutes past nine and 32 minutes to ten*.)

A3 **9 minutes** Charlie puts the seventh egg in the pan six minutes after he puts in the first egg. The seventh egg takes three minutes to cook, so he takes it out of the pan nine minutes after starting the whole operation.

A4 $\mathbf{\frac{81}{256}}$ After each hobbit eats his porridge, $\frac{3}{4}$ of what he started with remains. So after four have eaten, what remains is $\left(\frac{3}{4}\right)^4 = \frac{81}{256}$ of the original amount.

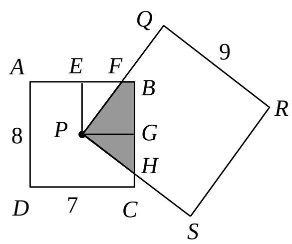
A5 **2** By comparing the dice at the top and bottom of the tower, it can be seen that the top face of the bottom die has five spots. So the face opposite, namely the face on which the tower stands, has two dots.

A6 **110°** The sum of the five interior angles of a pentagon is 540°, so the average size of these angles is 108°. As the sizes in degrees of the angles are consecutive whole numbers, they are 106°, 107°, 108°, 109°, 110°.

A7 $\mathbf{\frac{1}{2}}$ The visible end-face of the large cuboid consists of nine rectangles: five coloured white and four black. The face opposite this face also consists of nine rectangles: four coloured white and five black. So, between them, these two faces have equal numbers of white and black rectangles. Each of the other four faces of the large cuboid consists of twelve rectangles: six coloured white and six black. So the fraction of the surface area of the large cuboid which is coloured black is equal to the fraction which is coloured white, that is one half.

A8 **32** After the first pass, all the odd pegs have been knocked over and just the multiples of 2 remain standing. Likewise, after the next passes, first just the multiples of 4 remain, then those of 8, 16, 32. The final pass knocks down the only remaining peg, number 32.

A9 **18** Let the midpoints of AB and BC be E and G respectively and let F and H be the points shown.



Consider triangles PEF and PGH:
$\angle PEF = \angle PGH = 90°$; $\angle EPF = \angle GPH$, since both are equal to $90° - \angle FPG$;
$PE = PG$.

So the two triangles are congruent (AAS).

Applying Pythagoras' Theorem to triangle PEF: $PF^2 = PE^2 + EF^2 = 16 + 9 = 25$.

So PF has length 5 units.

As triangle PGH is congruent to triangle PEF, $GH = EF = 3$; $PH = PF = 5$.

So the perimeter of quadrilateral $PFBH$ is $(5 + 1 + 7 + 5)$ units = 18 units.

(*Note that the area of overlap between the two squares remains constant when square PQRS is rotated about point P, but the perimeter of the overlapping region changes*.)

A10 **37** The lengths of the sides of the triangles in the figure are shown in the table below:

Triangle	A	B	C	D	E	F	G	H
Length of side	1	1	1	2	2	3	4	5

The ninth triangle, I, will be placed alongside triangles H and D, so its sides will be 7 units long.

The tenth triangle, J, will be placed alongside triangles I and E, so its sides will be 9 units long.

The eleventh triangle, K, will be placed alongside triangles J and F, so its sides will be 12 units long.

As the spiral continues, each new triangle is placed alongside the triangle placed immediately before it in the sequence and the triangle placed five turns earlier than the new triangle. So, the twelfth triangle, L, will be placed alongside triangles K and G, giving it sides of length 16 units. The lengths of the sides of the next three triangles to be placed are shown in the table below.

Triangle number	Placed alongside	Length of side
13 (M)	L and H	16 + 5 = 21
14 (N)	M and I	21 + 7 = 28
15 (O)	N and J	28 + 9 = 37.

Section B

B1 Find four integers whose sum is 400 and such that the first integer is equal to twice the second integer, three times the third integer and four times the fourth integer.

Solution

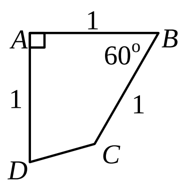
Let the first integer be x. Then the second, third and fourth integers are $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}$ respectively.

Therefore $x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 400$, that is $\frac{25x}{12} = 400$. So $x = 192$ and the four integers, in order, are 192, 96, 64, 48.

(*Note that a simpler solution is obtained by letting the first integer be* $12x$.)

B2 The diagram shows a quadrilateral $ABCD$ in which AB, BC and AD are all of length 1 unit, $\angle BAD$ is a right angle and $\angle ABC$ is $60°$.

Prove that $\angle BDC = 2 \times \angle DBC$.



Solution

Consider triangle ABC: it has two equal sides, AB and BC, and the angle between them is $60°$, so it is equilateral. Therefore $\angle ACB = \angle CAB = 60°$ and AC has length 1 unit.

Now consider triangle ACD: $\angle CAD = \angle DAB - \angle CAB = 90° - 60° = 30°$. Also, $CA = DA = 1$ unit, so $\angle ACD = \angle ADC = 75°$.

Next consider triangle ABD. It is a right-angled isosceles triangle, therefore $\angle ABD = \angle ADB = 45°$.

Finally, consider triangle BCD: $\angle BDC = \angle ADC - \angle ADB = 75° - 45° = 30°$; $\angle DBC = \angle ABC - \angle ABD = 60° - 45° = 15°$. So $\angle BDC = 2 \times \angle DBC$.

B3 (a) Yesterday evening, my journey home took 25% longer than usual.
By what percentage was my average speed reduced compared to normal?

(b) By what percentage would I need to increase my usual average speed in order for the journey to take 20% less time than usual?

Solution

(a) Let my distance from work be d and the normal journey time be t. Then my normal average speed is $\frac{d}{t}$.

Yesterday, my journey time was 25% longer than usual, that is $t \times \frac{5}{4}$.

Therefore yesterday's average speed was $d \div \frac{5t}{4} = d \times \frac{4}{5t} = \frac{4d}{5t} = \frac{4}{5} \times \frac{d}{t}$.

So my average speed yesterday was 80% of its usual value; hence it was reduced by 20%.

(b) If the journey is to take 20% less time than usual, then the new journey time will need to be $\frac{4t}{5}$.

Therefore the average speed will need to be $d \div \frac{4t}{5} = d \times \frac{5}{4t} = \frac{5}{4} \times \frac{d}{t}$.

So my usual average speed will need to be increased by 25% of its normal value.

B4 Find a rule which predicts exactly when the sum of five consecutive integers is divisible by 15.

Solution

Let the five consecutive integers be x, $x + 1$, $x + 2$, $x + 3$ and $x + 4$.

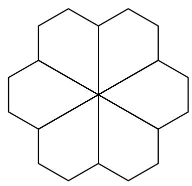
The sum of the numbers is $5x + 10 = 5(x + 2)$. This is a multiple of 15 if and only if 3 divides $x + 2$. So the sum of five consecutive integers is a multiple of 15 if and only if the third number is a multiple of 3.

(*A shorter proof is obtained by letting the consecutive integers be* $x - 2$, $x - 1$, x, $x + 1$ *and* $x + 2$. *Their sum is* $5x$, *which is clearly a multiple of* 5. *So it will be a multiple of* 15 *if and only if* x *is a multiple of* 3.)

B5 A window is constructed of six identical panes of glass. Each pane is a pentagon with two adjacent sides of length two units.

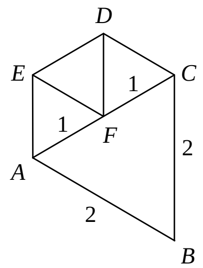
The other three sides of each pentagon, which are on the perimeter of the window, form half of the boundary of a regular hexagon

Calculate the exact area of glass in the window.



Solution

Let $ABCDE$ be one of the pentagonal panes which make up the window and let F be the midpoint of AC.



As six equal angles meet at point B, $\angle ABC = 360° \div 6 = 60°$.

So triangle ABC has equal sides, AB and BC, and the angle between them is 60°; so it is equilateral. Hence AC has length 2.

As F is the midpoint of AC, $\angle AFB = 90°$, so, by Pythagoras' Theorem: $BF = \sqrt{AB^2 - AF^2} = \sqrt{4 - 1} = \sqrt{3}$.

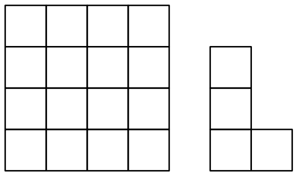
Therefore the area of triangle ABC is $\frac{1}{2} \times 2 \times \sqrt{3} = \sqrt{3}$.

As $ACDE$ forms half of a regular hexagon, its area may be divided up into three congruent equilateral triangles, each of side 1. Consider one of these triangles: it is similar to triangle ABC with sides in the ratio 1:2, so its area is one quarter of the area of triangle ABC, that is $\frac{\sqrt{3}}{4}$.

Therefore the area of pentagon *ABCDE* is $\sqrt{3} + 3 \times \frac{\sqrt{3}}{4} = \frac{7\sqrt{3}}{4}$.

So the exact area of glass in the window is $6 \times \frac{7\sqrt{3}}{4} = \frac{21\sqrt{3}}{2}$.

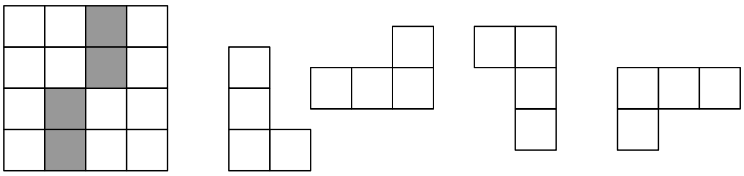
B6 We want to colour red some of the cells in the 4×4 grid shown so that wherever the L-shaped piece is placed on the grid it covers at least one red cell. The L-shaped piece may only cover complete cells, may be rotated, but may not be turned over and may not extend beyond the grid.



(a) Show that it is possible to achieve this by colouring exactly four cells red.

(b) Show that it is impossible to achieve this by colouring fewer than four cells red.

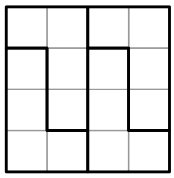
Solution



(a) The diagrams show the four different orientations in which the L-shaped piece may be placed on the grid. It can be seen that when four cells of the grid are coloured as shown, it is not possible to place the L-shaped piece, whatever its orientation, on the grid without at least one of the coloured cells being covered.

So by colouring four cells red it is possible to ensure that wherever the L-shaped piece is placed on the grid it covers at least one red cell.

(b) The diagram shows how four copies of the L-shaped piece may be placed on the 4×4 grid without overlap. If fewer than four cells of the grid are coloured red then at least one of the four copies will have none of its cells coloured, so it will be possible to place the L-shaped piece on the grid without it covering at least one red cell. So if fewer than four cells are coloured red, it is impossible to ensure that wherever the L-shaped piece is placed on the grid it covers at least one red cell.

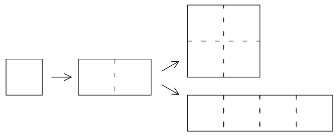


(*Please note that this is not the only diagram which may used show that the required task is impossible to achieve by colouring fewer than four cells red*.)

2008 Solutions

A1 **3** A two-digit prime cannot end with a 2, 5 or 8. So we need only check 23, 53 and 83, each of which is prime.

A2 **30 cm** As the square has a perimeter of 12 cm, it must have a side length of 3cm. Unfolding the square once gives a 6 cm × 3 cm rectangle.



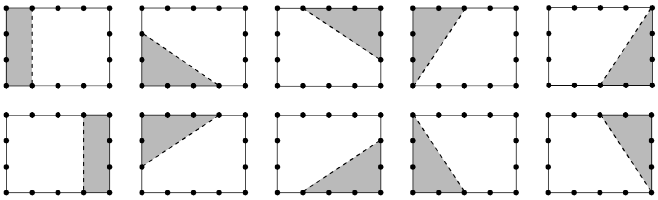
This rectangle can then be unfolded about either a long edge or a short edge resulting in a 6 cm × 6 cm square or a 12 cm × 3 cm rectangle. The perimeters of these are 24 cm and 30 cm respectively.

A3 **$x = 36$** Rearranging the equation, we have

$\frac{1}{x} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{12} - \frac{1}{18} = \frac{36 - 18 - 12 - 3 - 2}{36} = \frac{1}{36}$, and so $x = 36$.

A4 **9** The digits of a three-digit number whose product is 6 must be either two 1s and a 6, or a 1, 2 and 3. In the first case, there are three choices for the position of the 6; in the other case, there are three choices for the first digit and then two choices for the second. Hence, altogether, there are 9 such numbers: 116, 161, 611 and 123, 132, 213, 231, 312, 321.

A5 **10** The smaller area must be either a rectangle or a triangle of area 3. There are only two such rectangles, one at each end, as shown. The triangle needs to have base 2 and height 3 (or vice versa). There are two positions on each edge for the base. So we get eight triangles this way, as shown.



A6 **8** Factorising 1600 into the product of its prime factors gives $1600 = 2^6 \times 5^2$. The factors of 1600 are either 1 or the numbers less than or equal to 1600 whose prime factors are only 2 or 5. Of the latter, the square numbers are those where the powers of 2 and of 5 are even. So the square numbers that are factors of 1600 are $1, 2^2, 2^4, 2^6, 5^2, 2^2 \times 5^2, 2^4 \times 5^2$ and 1600 itself

[*Alternatively*: A factor of 1600 is a square number if and only if its square root is a factor of $\sqrt{1600} = 40$. There are eight factors of 40: 1, 2, 4, 5, 8, 10, 20 and 40.]

A7 **$x = 0$** As the bottom row and the diagonal running from 6 to x have the same sum, $x + 9$, the middle square must contain the number 3. Thus the remaining numbers in the top row must be $x + 2$ (from the second column) and $x + 1$ (from the diagonal).

$x + 1$	$x + 2$	**6**
	3	
$\boldsymbol{x}$	**4**	**5**

Considering the top row, we have $x + 1 + x + 2 + 6 = x + 9$, thus $x = 0$.

[*Alternatively*: It is well known that the magic sum of a 3×3 magic square is three times the middle number. Once the middle number is known it follows that $x = 0$.]

A8 **1** After Annie has taken her last 8 sweets, let x be the number of sweets that Clarrie, Lizzie and Annie have and s be the number of sweets left for Danni, where $0 < s < 8$. After Danni takes the remaining s sweets, she has a total of $x - 8 + s$.

Let the three girls each give e sweets to Danni, so that

$$x - e = x - 8 + s + 3e$$

which gives

$$4e = 8 - s.$$

Because $0 < 4e < 8$, we have $e = 1$.

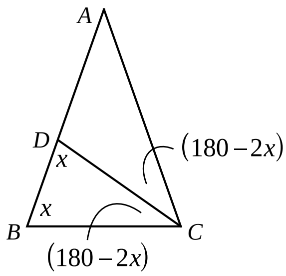
[*Alternatively*: Since the girls end up with equal numbers of sweets, the total number is a multiple of 4. They have all picked multiples of 8 except for the final number picked by Danni, and so she gets 4 sweets in her last turn. If each of the three other girls gives Danni 1 sweet, they all effectively gain 7 sweets in their last turn, whereas if they each gave more than 1 sweet, Danni would have more than them.]

A9 **108°** Triangle BCD is isosceles, so let $\angle DBC = \angle BDC = x°$.

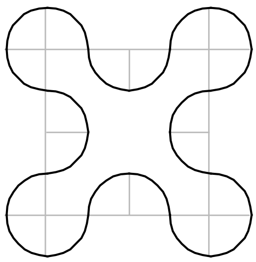
Then $\angle BCD = \angle ACD = (180 - 2x)°$.

Since triangle ABC is also isosceles, $\angle ACB = \angle ABC = x°$, and so $x = 180 - 2x + 180 - 2x$, and hence $x = 72$.

Thus $\angle CDA = (180 - x)° = 108°$.



A10 **(64 + 4π) cm²** Adding the construction lines shown, we can see that the area of the shape
= the area of the square + area of 4 quarter-circles
$= (4 \times 2)^2 + \pi \times 2^2 = (64 + 4\pi)\,\text{cm}^2$.



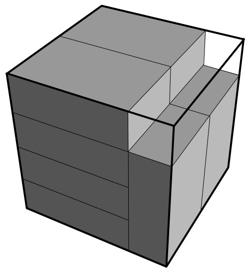
B1 Tamsin has a selection of cubical boxes whose internal dimensions are whole numbers of centimetres, that is, $1\text{ cm}\times 1\text{ cm}\times 1\text{ cm}$, $2\text{ cm}\times 2\text{ cm}\times 2\text{ cm}$, and so on.

What are the dimensions of the smallest of these boxes in which Tamsin could fit ten rectangular blocks each measuring $3\text{ cm}\times 2\text{ cm}\times 1\text{ cm}$ without the blocks extending outside the box?

Solution

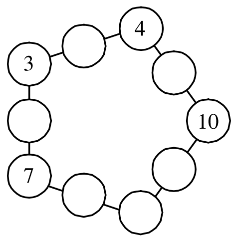
The total volume of 10 blocks, each $3\text{ cm}\times 2\text{ cm}\times 1\text{ cm}$, is $10\times 3\times 2\times 1 = 60\text{ cm}^3$.
The volumes of a 1 cm cube, a 2 cm cube and a 3 cm cube are 1 cm^3, 8 cm^3 and 27 cm^3 respectively so these cubes cannot contain the 10 blocks.

So the smallest *possible* cube is a 4 cm cube and the diagram on the right shows how such a cube can hold the 10 blocks.



B2 Each of the numbers from 1 to 10 is to be placed in the circles so that the sum of each line of three numbers is equal to T. Four numbers have already been entered.

Find all the possible values of T.



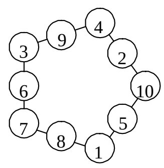
Solution

Let x be the number in the circle in the unfilled bottom corner of the pentagon.

All ten numbers are used once and only once and so the sum of the numbers in all ten circles is $1+2+3+4+5+6+7+8+9+10 = 55$.

Now, consider the numbers as they appear in the five lines of three circles. When these five sets of numbers are added, the corner numbers (3, 4, 7, 10, x) will be included twice so the sum is

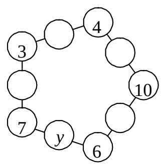
$$5T - 3 - 4 - 7 - 10 - x = 5T - x - 24.$$



Equating the two sums, we have $5T - x - 24 = 55$ which gives $5T = 79 + x$.

Since T is an integer, $5T$ is a multiple of 5, which means that x must be either 1 or 6.

If $x = 1$ then $T = 16$, while if $x = 6$, $T = 17$.



The first diagram shows that $x = 1$, $T = 16$ is a possibility, while in the second diagram, $x = 6$, $T = 17$ forces $y = 4$, repeating the 4 in the top corner.

Thus the only possible value of T is 16.

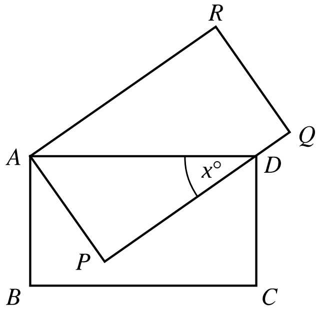
[*Alternatively*: Let the number in the circle between 4 and 10 be a. Then the total for each line T is $a + 14$, and so the number in the circle between 3 and 4 is $a + 7$. Yet this is between 1 and 10 and cannot be 7 or 10, and so $a = 1$ or $a = 2$.

If we try $a = 1$, we get $T = 15$, but there is no way of completing the other line including the 10, without reusing either the 3 or the 4.

If we try $a = 2$, we obtain the solution shown above, and hence $T = 16$ is the only possibility.]

B3 In the diagram *ABCD* and *APQR* are congruent rectangles. The side *PQ* passes through the point *D* and $\angle PDA = x°$.

Find an expression for $\angle DRQ$ in terms of *x*.

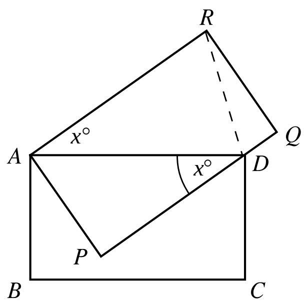


Solution

Since *APQR* is a rectangle, the lines *AR* and *PQ* are parallel. Thus $\angle PDA = \angle DAR$ (alternate angles) so $\angle DAR = x°$.

Since $DA = RA$, triangle *DAR* is isosceles and so $\angle ARD = \angle ADR = \frac{1}{2}(180 - x)° = (90 - \frac{1}{2}x)°$.

Hence, since $\angle ARQ = 90°$, $\angle DRQ = \frac{1}{2}x°$.



B4 For each positive two-digit number, Jack subtracts the units digit from the tens digit; for example, the number 34 gives $3 - 4 = -1$.

What is the sum of all his results?

Solution

The two-digit numbers run from 10 to 99.

We may categorise these numbers into the following four sets:

P = {the numbers *ab* where *a* and *b* are strictly positive and $a < b$}

Q = {the numbers *ab* where *a* and *b* are strictly positive and $b < a$}

R = {the palindromes *aa*}

S = {the numbers *ab* where $a > 0$ and $b = 0$}.

For numbers '*ab*', the units digit subtracted from the tens digit is $a - b$.

The result $a - b$ from each of the numbers in set P can be matched with the result $b - a$ from each corresponding number in the set Q, giving a total of $(a - b) + (b - a) = 0$.

For each of the numbers in set R the result is $a - a$ which is 0.

Finally, when we consider all of the numbers in set S, the result is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$. So the total is 45.

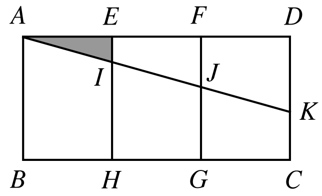
[*Alternatively*: The units digits for all the two-digit integers comprise the numbers 0, 1, ..., 9 each 9 times (coming after 1, 2, ..., 9).

The tens digits are 1, 2, ..., 8, 9, each coming 10 times (before 0, 1, ..., 9).

The difference is thus 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45.]

B5 In the diagram, the rectangle $ABCD$ is divided into three congruent rectangles. The line segment JK divides $CDFG$ into two parts of equal area.

What is the area of triangle AEI as a fraction of the area of $ABCD$?



Solution

Let a = the area of AEI.

We have $AF = 2 \times AE$ and $AD = 3 \times AE$. Also the triangles AEI, AFJ and ADK are similar since $\angle A$ is common and the angles at E, F and D are right-angles. Hence the area of $ADK = 9a$, the area of $AFJ = 4a$ and so the area of $DFJK = 9a - 4a = 5a$.

So the area of $DFGC = 2 \times 5a = 10a$ and the area of $ABCD = 3 \times 10a = 30a$.

Thus $\dfrac{\text{area } AEI}{\text{area } ABCD} = \dfrac{1}{30}$.

B6 In a sequence of positive integers, each term is larger than the previous term. Also, after the first two terms, each term is the sum of the previous two terms.

The eighth term of the sequence is 390. What is the ninth term?

Solution

Let the first two terms be the positive integers a and b. So the first nine terms of the sequence are:

$$a,\ b,\ a + b,\ a + 2b,\ 2a + 3b,\ 3a + 5b,\ 5a + 8b,\ 8a + 13b,\ 13a + 21b.$$

Thus $$8a + 13b = 390$$

giving $$8a = 390 - 13b = 13(30 - b). \qquad (*)$$

So $8a$ is a multiple of 13, and hence a itself is a multiple of 13.

Since $8a = 390 - 13b$, if $a \geqslant 26$ then $390 - 13b \geqslant 8 \times 26$, i.e. $13b \leqslant 390 - 208 = 182$ which means $b \leqslant 14 < a$ and the sequence would not be increasing.

So $a = 13$ and $13b = 390 - 13 \times 8$ giving $b = 22$.

Thus the ninth term is $13a + 21b = 13 \times 13 + 21 \times 22 = 169 + 462 = 631$.

[*Alternative ending*: From the starred equation, $30 - b$ is a multiple of 8, and hence could be 0, 8, 16 or 24. This means that b could be 30, 22, 14 or 6, from which the corresponding values of a are 0, 13, 26 and 39.

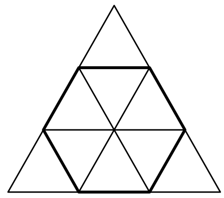
We know that $0 < a < b$, so the only possibility here is $a = 13$ and $b = 22$.

Thus the ninth term is $13a + 21b = 13 \times 13 + 21 \times 22 = 169 + 462 = 631$.]

2009 Solutions

A1 **40081** $200^2 + 9^2 = 40000 + 81 = 40081$.

A2 **40 cm²** Since the hexagon is regular, it has interior angles of $120°$ and can be dissected into six congruent triangles. The small triangles have three angles of $60°$ and are therefore equilateral with side equal to that of the hexagon. The three triangles inside the large triangle but outside the hexagon are also equilateral with the same side length as the hexagon. So the area of the hexagon is $\frac{6}{9}\left(= \frac{2}{3}\right)$ of the area of the original triangle.



A3 **17** It is clear that each of a, b and c must be less than or equal to 10. A brief inspection will show that the only combination of different square numbers which total 121 is $81 + 36 + 4$.

More formally, the problem can be analysed by considering the remainders after dividing the square numbers less than 121 (1, 4, 9, 16, 25, 36, 49, 64, 81 and 100) by three: the remainders are 1, 1, 0, 1, 1, 0, 1, 1, 0 and 1.

When 121 is divided by 3, the remainder is 1. Therefore $a^2 + b^2 + c^2$ must also leave a remainder of 1. Now we can deduce that two of the three squares must leave a remainder of 0 and so be multiples of 3. There are three square numbers below 121 which are multiples of three: 9, 36 and 81. Checking these, we see that 81 and 36 are the only pair to have a sum which differs from 121 by a perfect square, namely 4. So $a + b + c = 9 + 6 + 2 = 17$.

A4 **0** The sum of the first two numbers and the last two numbers is $1004 + 1005 = 2009$. This counts the middle number twice. But the sum of all three numbers is 2009, so the middle number is 0. Hence the product of all three numbers is 0.

[*Alternatively*: Let the three numbers be a, b and c.

We have $$a + b = 1004,$$

$$b + c = 1005$$

and $$a + b + c = 2009.$$

Adding the first two equations gives

$$a + 2b + c = 2009$$

and subtracting the third equation from this gives

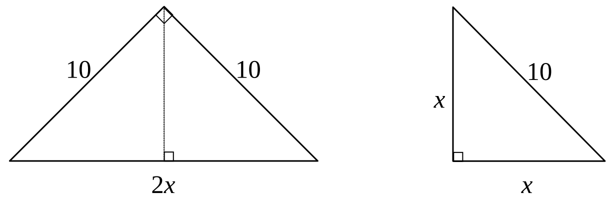
$$b = 0.$$

Thus the product $abc = 0$.]

A5 **$18\frac{1}{3}$** The volume of petrol that Andrea put in, as a fraction of the volume of the tank, is the difference between $\frac{2}{3}$ and $\frac{1}{4}$, which is $\frac{5}{12}$. So she put in $\frac{5}{12}$ of 44 litres and $\frac{5}{12} \times 44 = \frac{5 \times 44}{12} = \frac{5 \times 11}{3} = \frac{55}{3} = 18\frac{1}{3}$.

A6 **10 cm** Since the original triangle is isosceles and right-angled, folding it produces a smaller triangle, also isosceles and right-angled. By Pythagoras' Theorem, the hypotenuse of the original triangle is $\sqrt{200} = 10\sqrt{2}$ cm. Hence the difference between the perimeters of the two triangles is $\left(10 + 10 + 10\sqrt{2}\right) - \left(5\sqrt{2} + 5\sqrt{2} + 10\right) = 10$ cm.

Alternatively: Let the length of the shorter sides of the new triangle be x cm, shown below. Then the perimeter of the original triangle is $(20 + 2x)$ cm and the perimeter of the new triangle is $(10 + 2x)$ cm. Hence the difference between the perimeters of the two triangles is 10 cm.

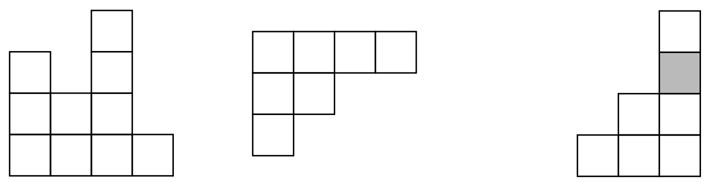


A7 **11** As a fraction of 30 minutes, 30 seconds is $\frac{1}{60}$. So we are considering fractions with a denominator of 60. To obtain a fraction of the required form, the numerator must be
a factor of 60 (and less than 60). The numerator can therefore be 1, 2, 3, 4, 5, 6, 10, 12, 15, 20 or 30.

A8 **22π cm** The length of a semicircular arc of radius r is πr and so the total perimeter is $\left(2 \times (1 + 2 + 4) + 8\right)\pi = 22\pi$ cm.

A9 **133** After every two numbers, one is omitted. Because $89 = 2 \times 44 + 1$, there must be 44 page numbers missing and so the number on the last page is $89 + 44 = 133$.

A10 **5000 cm^2** Views from the front and back, the top and bottom, and the two sides are as shown below:



2 lots of 10 faces | 2 lots of 7 faces | 2 lots of 8 faces (one hidden from each side)

Each square face has a surface area of 100 cm^2. Hence the total surface area of Gill’s shape is $(20 + 14 + 16) \times 100 \text{ cm}^2 = 5000 \text{ cm}^2$.

B1 In 2007 Alphonse grew twice the number of grapes that Pierre did. In 2008 Pierre grew twice the number of grapes that Alphonse did. Over the two years Alphonse grew 49 000 grapes, which was 7600 less than Pierre. How many grapes did Alphonse grow in 2007?

Solution

Suppose Pierre grew p grapes in 2007. Then, in 2007, Alphonse grew $2p$ grapes.
Thus, in 2008, Alphonse grew $49\,000 - 2p$ and so Pierre grew $98\,000 - 4p$.
Over the two years, the number of grapes Pierre grew was

$$p + (98\,000 - 4p) = 49\,000 + 7600$$

so

$$41\,400 = 3p$$

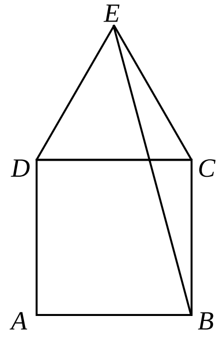
and

$$p = 13\,800.$$

Hence, in 2007, Alphonse grew $2 \times 13\,800 = 27\,600$ grapes.

B2 $ABCD$ is a square. The point E is outside the square so that CDE is an equilateral triangle. Find angle BED.

Solution



Since $ABCD$ is a square, $\angle BCD = 90°$; and since CDE is an equilateral triangle, $\angle DCE = 60°$.

Thus $\angle BCE = \angle BCD + \angle DCE = 90° + 60° = 150°$.

Because CDE is an equilateral triangle, $EC = DC$ and also, because $ABCD$ is a square, $DC = CB$. Hence $EC = CB$ and ECB is an isosceles triangle.

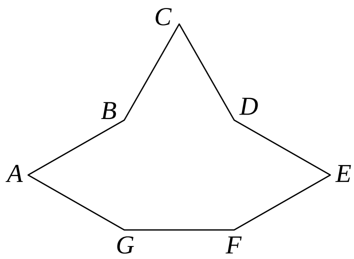
So $\angle CEB = \angle CBE = \frac{1}{2}(180 - 150)° = 15°$, and hence $\angle BED = \angle CED - \angle CEB = 60° - 15° = 45°$.

B3 Tom left a motorway service station and travelled towards Glasgow at a steady speed of 60 mph. Tim left the same service station 10 minutes after Tom and travelled in the same direction at a steady speed, overtaking Tom after a further 1 hour 40 minutes. At what speed did Tim travel?

Solution

Tom travels for 10 minutes longer than Tim, a time of 1 hour and 50 minutes.
Travelling at a speed of 60 mph (or 1 mile per minute), Tom travels a distance of 110 miles.
Tim travelled the same distance in 1 hour and 40 minutes ($1\frac{2}{3}$ hours),
so his speed, in mph, was $110 \div 1\frac{2}{3} = 110 \times \frac{3}{5} = 22 \times 3 = 66$ mph.

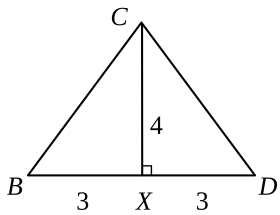
B4 The diagram shows a polygon *ABCDEFG*, in which $FG = 6$ and $GA = AB = BC = CD = DE = EF$. Also *BDFG* is a square. The area of the whole polygon is exactly twice the area of *BDFG*. Find the length of the perimeter of the polygon.



Solution

The area of square *BDFG* is $6 \times 6 = 36$ square units.
So the total area of the three triangles *ABG*, *BCD* and *DEF* is also 36 square units.
These three triangles are congruent and so each has an area of 12 square units.

The area of each triangle is $\frac{1}{2} \times$ base $\times$ height and the base is 6 units and hence we have $\frac{1}{2} \times 6 \times \text{height} = 12$, so the height is 4 units.

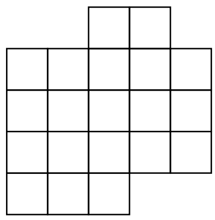


Let *X* be the midpoint of *BD* Then *CX* is perpendicular to the base *BD* (since *BCD* is an isosceles triangle).

By Pythagoras' Theorem, $BC = \sqrt{3^2 + 4^2} = 5$ units.

Therefore the perimeter of *ABCDEFG* is $6 \times 5 + 6 = 36$ units.

B5 An ant wishes to make a circuit of the board shown, visiting each square exactly once and returning to the starting square. At each step the ant moves to an adjacent square across an edge. Two circuits are considered to be the same if the first follows the same path as the second but either starts at a different square or follows the same path in reverse. How many such circuits are possible?



Solution

From Figure 1, we see that there is only one possible route through each of the corner squares, shown shaded, so the ant's path through these squares is as indicated.
There is then only one possible route through square X, giving Figure 2.

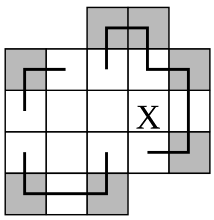


Figure 1

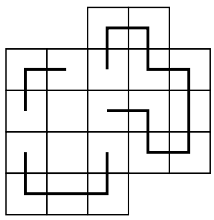


Figure 2

Now consider the shaded square in Figure 3.

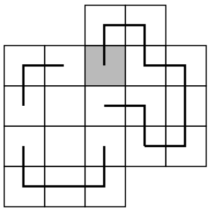


Figure 3

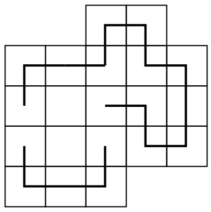


Figure 4

If the path joined this square to the square below, then a closed loop would be formed and the ant could not complete a circuit of the board. Hence the path joins the shaded square to the square on its left (Figure 4).

There are now only two squares which the ant's path has not visited. If a path through all the squares did not join these two, then two loops would be formed instead of a single circuit. We deduce that the path joins these two squares and then there are only two ways of completing the path, as shown in Figure 5.

Figure 5

B6 I want to choose a list of n different numbers from the first 20 positive integers so that no two of my numbers differ by 5. What is the largest value of n for which this is possible? How many different lists are there with this many numbers?

Solution

Any such list contains at most two numbers from the set {1, 6, 11, 16}, at most two numbers from the set {2, 7, 12, 17}, and likewise from each of the sets {3, 8, 13, 18}, {4, 9, 14, 19} and {5, 10, 15, 20}. Hence there are at most $5 \times 2 = 10$ numbers altogether. The list of ten numbers, 1, 2, 3, 4, 5, 11, 12, 13, 14, 15 shows that a selection is indeed possible.

From each of these sets of four numbers of the form $\{a, a+5, a+10, a+15\}$, there are three pairs which do not differ by 5, namely $(a, a+10\}$, $\{a, a+15\}$ and $\{a+5, a+15\}$. Since we are choosing a pair from each of five such sets, there will be $3^5 = 243$ different lists.

2010 Solutions

A1 **55** $\frac{1}{1}+\frac{2}{\frac{1}{2}}+\frac{3}{\frac{1}{3}}+\frac{4}{\frac{1}{4}}+\frac{5}{\frac{1}{5}} = 1 + 2 \times 2 + 3 \times 3 + 4 \times 4 + 5 \times 5 = 1 + 4 + 9 + 16 + 25 = 55.$

A2 **3 : 8** Since y is common to both ratios, we change the ratios so that $x : y = 1 : 2 = 3 : 6$ and $y : z = 3 : 4 = 6 : 8$. Then we have $x : z = 3 : 8$.

A3 **14** We can note that $20^{10} = (2 \times 10)^{10} = 2^{10} \times 10^{10}$. Since $2^{10} = 1024$ has 4 digits, and multiplying by 10^{10} adds 10 zeros to the end, Tom writes down 14 digits.

A4 $\mathbf{\frac{1}{3}}$ The table shows the ways in which the monkeys (B, H and L) can select the hats. Let the hats of B, H and L be b, h and l respectively.

Monkeys			
B	H	L	
b	h	l	
b	l	h	
h	b	l	
h	l	b	*
l	h	b	
l	b	h	*

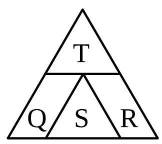
None of the monkeys have the same hat as when they arrived in only two of the six ways (*), hence the required probability is

$$\frac{2}{6} = \frac{1}{3}.$$

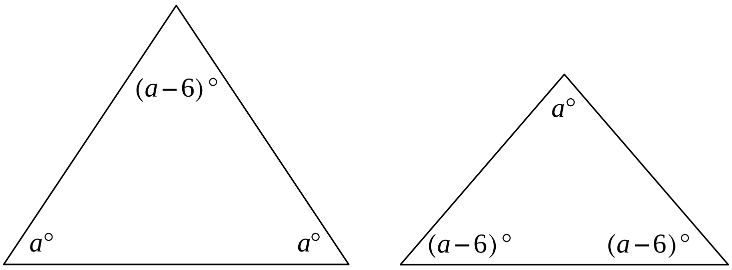
[*Alternatively*: There are $3 \times 2 \times 1 = 6$ possible ways to choose the three hats. There are two hats that B could choose. If B chose h, then L would have to choose b and H would have to choose l. If B chose l, then H would have to choose b and L would have to choose h. So once B has chosen his hat the other two are fixed. So there are just the two possible alternatives out of the six ways. So the probability is $\frac{2}{6} = \frac{1}{3}$.]

A5 **2010** Let the two numbers be a and b, where $a > b$. Then we have $a + b = 97$ and $a - b = 37$. Hence $2a = 134$ and therefore $a = 67$ and $b = 30$. The product of 67 and 30 is 2010.

A6 **1 : 1** Let us call the large triangle P. Since triangles T and S are congruent, they have the same height, which is half the height of P. Thus the area of each of T and S is a quarter of the area of P. Therefore parallelograms Q and R together form the other half and thus each occupies a quarter of P. So R and T are equal in area.



A7 **64°** Let the largest angle be $a°$, whence the smallest angle is $(a - 6)°$. There are two possibilities, shown in the diagrams below.



In the first we have $3a - 6 = 180$, so $a = 62$.
In the second we have $3a - 12 = 180$, so $a = 64$.
Thus the largest possible angle in such a triangle is $64°$.

A8 **13** To create a closed loop, one must use  at one end and  at the other.

Let us assume that the loop starts with  (turned this way) and ends with the , in some orientation (,  or ).
There is just 1 loop that uses only these tiles. If one tile is put between them, there are two ways in which each of the other two tiles can connect them ( or  and  or ). So there are 4 loops with three tiles.

Using all four tiles, there are two orders in which  and  can be placed, and, there are two possible orientations for each of these tiles, making $2 \times 2 \times 2 = 8$ ways in all. Hence there are $1 + 4 + 8 = 13$ possible loops altogether.

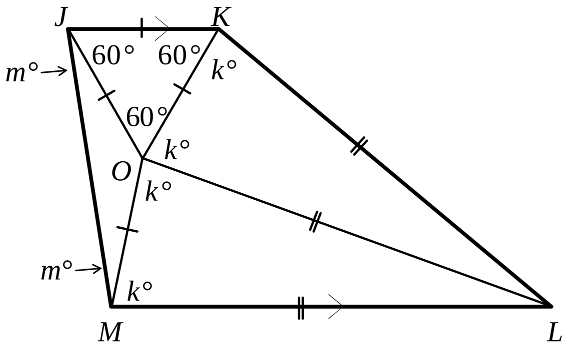
A9 **2 hours** Since the lowest common multiple of 6, 10 and 15 is 30, we can say that in 30 minutes Abbie writes $7 \times \frac{30}{6} = 35$ cards, Betty writes $18 \times \frac{30}{10} = 54$ cards, and Clara writes $23 \times \frac{30}{15} = 46$ cards. So together they write $35 + 54 + 46 = 135$ cards in half an hour. Thus the time taken to write 540 cards is $\frac{540}{135}$ = 4 half-hours = 2 hours.

A10 20° Since triangle JKO is equilateral, $\angle JOK = \angle KJO = \angle JKO = 60°$.

Let $\angle JMO = m°$. Then, since JMO is an isosceles triangle, $\angle MJO = m°$ and $\angle JOM = (180 - 2m)°$.

Let $\angle OKL = k°$ and so, since KLO is an isosceles triangle, $\angle LOK = k°$.

Triangles KLO and OLM are congruent (SSS), and so $\angle MOL = \angle OML = k°$.



Now taking angles at point O, we have $180 - 2m + 60 + 2k = 360$, whence $k = m + 60$.

Since JK is parallel to ML, $\angle KJM + \angle JML = 180°$ and so $(60 + m) + (m + k) = 180$.

Hence $180 = 60 + 2m + k = 60 + 2m + m + 60 = 3m + 120$, so $m = 20$, i.e. $\angle JMO = 20°$.

B1 In a sequence of six numbers, every term after the second term is the sum of the previous two terms. Also, the last term is four times the first term, and the sum of all six terms is 13.

What is the first term?

Solution

Let the first and second terms be a and b respectively. Then we derive the sequence

$$a,\ b,\ a + b,\ a + 2b,\ 2a + 3b,\ 3a + 5b.$$

We know that the last term is four times the first term, so $3a + 5b = 4a$. Therefore $a = 5b$ and so the sequence is

$$5b,\ b,\ 6b,\ 7b,\ 13b,\ 20b.$$

The sum of these is 13, so $52b = 13$, $b = \frac{13}{52} = \frac{1}{4}$ and $a = 5 \times \frac{1}{4} = \frac{5}{4}$. Thus the first term is $1\frac{1}{4}$.

B2 The eight-digit number "*ppppqqqq*", where p and q are digits, is a multiple of 45.

What are the possible values of p?

Solution

It might be argued that there is a trivial solution where $p = q = 0$. It is, however, usual to assume that numbers do not begin with zeros and so we shall proceed assuming that $p \neq 0$.

We first observe that every multiple of 45 is a multiple of both 5 and 9, and also that p and q are single-digit integers. Applying the usual rules of divisibility by 5 and 9 to the number *ppppqqqq* we deduce that $q = 0$ or $q = 5$ and that $4p + 4q$ is a multiple of 9.

In the case $q = 0$, $4p$ is a multiple of 9, hence $p = 9$.

In the case $q = 5$, $4p + 20 = 4(p + 5)$ is a multiple of 9. Therefore $p + 5$ is a multiple of 9. Hence $p = 4$.

(Thus there are two possible numbers: 99 990 000 and 44 445 555.)

B3 Jack and Jill went up a hill. They started at the same time, but Jack arrived at the top one-and-a-half hours before Jill. On the way down, Jill calculated that, if she had walked 50% faster and Jack had walked 50% slower, then they would have arrived at the top of the hill at the same time.

How long did Jill actually take to walk up to the top of the hill?

Solution

Let t be the number of hours that Jill took to the top of the hill.

So the time taken by Jack was $\left(t - 1\frac{1}{2}\right)$ hours.

If Jack had walked 50% more slowly, he would have taken twice as long, ie. $(2t - 3)$ hours.

If Jill had walked 50% faster, she would have taken $\frac{2}{3}$ of the time, ie. $\frac{2}{3}t$ hours.

So we know that $\frac{2}{3}t = 2t - 3$, whence $2t = 6t - 9$ and so $t = \frac{9}{4} = 2\frac{1}{4}$.

Hence Jill took $2\frac{1}{4}$ hours.

B4 The solution to each clue of this crossnumber is a two-digit number, not beginning with zero.

In how many different ways can the crossnumber be completed correctly?

Clues
Across
1. A triangular number
3. A triangular number

Down
1. A square number
2. A multiple of 5

1	2
3	

Solution

We start by listing the two-digit triangular numbers and two-digit square numbers:

triangular numbers:	10, 15, 21, 28, 36, 45, 55, 66, 78, 91
square numbers:	16, 25, 36, 49, 64, 81.

Since 2 Down is a multiple of 5, it ends in either 0 or 5.

Hence 3 Across ends in either 0 or 5 and there are four such triangular numbers: 10, 15, 45, and 55. In each case there is only one possible square number at 1 Down, as shown in the following figures:

8	
1	0

(a)

8	
1	5

(b)

6	
4	5

(c)

2	
5	5

(d)

Now consider 1 Across, a triangular number. In (a) and (b), there is no two-digit triangular number whose first digit is 8, and hence we can rule out cases (a) and (b).

In (c), the only triangular number whose first digit is 6 is 66. In (d), there are two triangular numbers whose first digit is 2, namely 21 and 28.

Therefore there are three different ways in which the crossnumber can be completed:

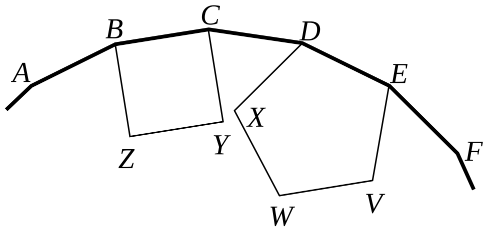
6	6
4	5

2	1
5	5

2	8
5	5

B5 The diagram shows part of a regular 20-sided polygon (an icosagon) $ABCDEF\ldots$, a square $BCYZ$ and a regular pentagon $DEVWX$.

Show that the vertex X lies on the line DY.

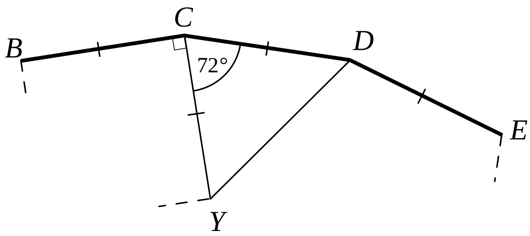


Solution

Considering the interior angles of the square, the regular pentagon, and the regular icosagon, $\angle BCY = 90°$, $\angle EDX = \left(180 - \frac{360}{5}\right)° = 108°$ and $\angle BCD = \left(180 - \frac{360}{20}\right)° = 162°$.

Now $\angle DCY = (162 - 90)° = 72°$ and also $\angle CDX = (162 - 108)° = 54°$.

Now consider triangle CDY.



Since the icosagon is regular, $BC = CD$ and, as $BCYZ$ is a square, $BC = CY$.

Therefore $CD = CY$ and CDY is an isosceles triangle.

Hence $\angle CDY = \frac{1}{2}(180 - 72)° = 54°$.

However, as observed above, $\angle CDX = 54°$ and so $\angle CDX = \angle CDY$.

Thus we can conclude that point X lies on the line DY.

B6 Sam has put sweets in five jars in such a way that no jar is empty and no two jars contain the same number of sweets. Also, any three jars contain more sweets in total than the total of the remaining two jars.

What is the smallest possible number of sweets altogether in the five jars?

Solution

Let the number of sweets in the five jars be a, b, c, d and e, where $a < b < c < d < e$. Since $d > c > b$, and b, c and d are integers, $d \geqslant b + 2$. Similarly $e \geqslant c + 2$.

Now, since any three jars contain more sweets in total than the total of the remaining two jars, in particular $a + b + c > d + e$, and so $a + b + c > b + 2 + c + 2$, hence $a > 4$.

Try $a = 5$. The smallest possible values of the other numbers are 6, 7, 8 and 9, which give a total of 35. Because 5, 6 and 7, the three smallest numbers, give a total of 18, which is over half of 35, any other selection of three of these numbers will have a total greater than that of the remaining two numbers.

Thus the smallest total is 35.